Multiverses with more than two Modal Logics of Forcing

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The syntax of modal logic is the one of propositional logic together with symbols

- $\Box$ 'necessary'
- $\Diamond$ 'possible'

We view $\Diamond$ as an abbreviation for $\neg\Box\neg$.

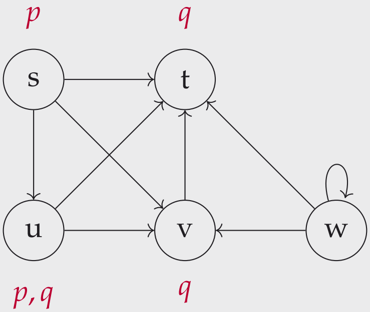
Definition

A *Kripke frame* $(F, \leq)$ consists of a set of worlds F and an accessibility relation $\leq$.
A *Kripke model* $(F, \leq, \|\cdot\|)$ consists of a frame and a valuation $\|\cdot\| : \mathrm{Prop} \to \wp(F)$.
We extend $\|\cdot\|$ to arbitrary modal formulas by

$$
\begin{aligned}
\|\neg\psi\| &= F \setminus \|\psi\| \\
\|\psi \wedge \chi\| &= \|\psi\| \cap \|\chi\| \\
\|\Box\psi\| &= \{w \in F \mid {\leq}[w] \subseteq \|\psi\|\} = \{w \in F \mid \forall v \geq w\,(v \in \|\psi\|)\}
\end{aligned}
$$

In a Kripke model, $(F, \leq, \|\cdot\|), w \models \psi$ ('ψ is true at w') iff $w \in \|\psi\|$.
In a Kripke frame, $(F, \leq), w \models \psi$ ('ψ is valid at w') iff $w \in \|\psi\|$ for every valuation $\|\cdot\|$.
$(F, \leq) \models \psi$ ('ψ is valid') iff $(F, \leq), w \models \psi$ for every $w \in F$.

Example



$(F, \leq, \| \cdot \|), s \vDash p \wedge \Diamond p \wedge \neg \Box p \wedge \Box q$

$(F, \leq, \| \cdot \|), w \nvDash \Box q$

$(F, \leq), s \vDash \Box p \rightarrow \Box\Box p$

$(F, \leq) \vDash \Box p \rightarrow \Box\Box p$

$(F, \leq), w \vDash \Box p \rightarrow p$

$(F, \leq) \nvDash \Box p \rightarrow p$

$(F, \leq), t \vDash \Box p$

Definition

A set of modal formulas is a *normal modal logic* if it contains all propositional tautologies,

$$(\mathrm{K}) = \Box(p \to q) \to (\Box p \to \Box q),$$

and is closed under

$$\frac{\psi \to \chi \qquad \psi}{\chi}\,\mathrm{MP} \qquad \frac{\psi(p_1, \dots, p_n)}{\psi(\chi_1, \dots, \chi_n)}\,\mathrm{US} \qquad \frac{\psi}{\Box\psi}\,\mathrm{Nec}$$

K is the smallest normal modal logic.

Lemma

$\mathrm{Log}(F, \leq) = \{\psi \mid (F, \leq) \models \psi\}$ *is normal for any frame* $(F, \leq)$.
However, $\{\psi \mid (F, leq), w \models \psi\}$ *is not always closed under* Nec.

Lemma

Let $\mathcal{C}$ *be the class of all Kripke frames.*
Then $\mathsf{K} = \mathrm{Log}(\mathcal{C}) = \bigcap_{(F,\leq)\in\mathcal{C}} \mathrm{Log}(F, \leq)$*, i.e.* K *is sound and complete w.r.t.* $\mathcal{C}$.

Definition

Let $L + \psi$ be the smallest normal modal logic containing L and φ.
$\mathsf{S4} = \mathsf{K} + \Box p \to p + \Box p \to \Box\Box p$
$\mathsf{S4.2} = \mathsf{S4} + \Diamond\Box p \to \Box\Diamond p$
$\mathsf{S5} = \mathsf{S4} + \Diamond\Box p \to p$

Lemma

S4 *is sound and complete w.r.t. (finite) reflexive and transitive frames.*
S4.2 *is sound and complete w.r.t. (finite) reflexive, transitive, and directed*[1] *frames.*
S5 *is sound and complete w.r.t. (finite) reflexive, transitive, and symmetric frames.*

[1] $x \leq y$ and $x \leq z \Rightarrow \exists w\,(y \leq w$ and $z \leq w)$

Nec is sometimes easy

Lemma

In a reflexive and transitive frame, $(F, \leq), w \models \Box\varphi$ *implies* $(F, \leq), w \models \mathsf{S4} + \varphi$.

Proof.

We show by induction on proofs that $(F, \leq), v \models \mathsf{S4} + \varphi$ for all $v \geq w$. By reflexivity, this implies $(F, \leq), w \models \mathsf{S4} + \varphi$.
If $\psi \in \mathsf{S4} \cup \{\varphi\}$, there is nothing to show.
Since $\{\psi \mid (F, \leq), v \models \psi\}$ is always closed under MP and US, these cases are easy.
Assume $(F, \leq), u \models \psi$ for all $u \geq w$, i.e. $(F, \leq), w \models \Box\psi$, and let $v \geq w$. By transitivity, $(F, \leq), v \models \Box\psi$. $\Box$

Definition

Let M be a countable transitive model of ZFC and φ a sentence of set theory.
$M \models \Box\varphi$ iff $\Vdash_{\mathbb{P}} \varphi$ for every forcing poset $\mathbb{P} \in M$.
The *modal logic of forcing* of M is

$$\mathrm{MLF}(M) = \{\psi(q_1, \ldots, q_n) \mid M \models \psi(\varphi_1, \ldots, \varphi_n) \text{ for all } \varphi_1, \ldots, \varphi_n \in \mathcal{L}_\in\}.$$

The *modal logic of forcing* is

$$\mathrm{MLF} = \bigcap_{M \models \mathrm{ZFC} \text{ countable}} \mathrm{MLF}(M).$$

By the Truth Lemma, we can think of MLF as the modal formulas valid on the Kripke frame of models of ZFC with $M \leq N$ iff N is a generic extension of M. Then $\mathrm{MLF}(M)$ is the set of modal formulas valid at M.

Theorem (Hamkins and Löwe [4])

S4.2 $\subseteq$ MLF.

Proof.

The 'frame' is reflexive, transitive, and directed.

Reflexivity	M is a generic extension of itself
Transitivity	iterated forcing $\mathbb{P} * \mathbb{Q}$
Directedness	product forcing $\mathbb{P} \times \mathbb{Q}$

Hence S4.2 is valid on it. □

Definition

A sentence φ of set theory is a *switch* in M if $M \models \Box\Diamond\varphi \wedge \Box\Diamond\neg\varphi$.
A sentence φ of set theory is a *button* in M if $M \models \Diamond\Box\varphi$. It is unpushed if $M \not\models \varphi$.

Example

CH is a switch in any model.
In L, "ω_{n+1}^L is not a cardinal" is an unpushed button.

Definition

A family of buttons b_i and switches s_j is independent in M if none of the buttons is pushed and in any generic extension of M, any button can be pushed and any switch can be switched without affecting the others.

Lemma

S4.2 *is complete w.r.t. the class of finite reflexive, transitive, and directed frames.*

Definition

A frame $(F, \leq)$ is a *pre-lattice* if $(F/{\equiv}, \leq)$ is a lattice, where $v \equiv w$ iff $v \leq w \leq v$.

Lemma

S4.2 *is complete w.r.t. the class of finite pre-lattices.*

Pre-Lattices II

Proof.

Let $(F, \leq)$ be finite reflexive, transitive, and directed frame with $(F, \leq), w \not\models \psi$.
W.l.o.g. $w \leq v$ for all $v \in M$. Directedness implies that there is a top cluster $[t]$.
Unravel the frame of clusters at $[w]$, but merge all copies of $[t]$.
The unravelling is a finite pre-lattice:

$$[u] \wedge [v] = \begin{cases} \min([u], [v]) & \text{if } u \text{ and } v \text{ are comparable} \\ [w] & \text{otherwise} \end{cases}$$

$$[u] \vee [v] = \begin{cases} \max([u], [v]) & \text{if } u \text{ and } v \text{ are comparable} \\ [t] & \text{otherwise.} \end{cases}$$

Since the two frames are bisimilar, ψ is not valid in the unravelling either. □

Lemma

Let $(F, \leq, \|\cdot\|)$ be a Kripke model on a finite pre-lattice and $w \in F$. Let $M \models \mathsf{ZFC}$ have an independent family of infinitely many buttons b_i and infinitely many switches s_j. Then there are $\varphi_i \in \mathcal{L}_\in$ such that $(F, \leq, \|\cdot\|), w \models \psi(q_1, \ldots, q_n)$ iff $M \models \psi(\varphi_1, \ldots, \varphi_n)$ for all modal formulas ψ.

Theorem (Hamkins and Löwe [4])

$\mathrm{MLF} \subseteq \mathrm{MLF}(L) \subseteq \mathsf{S4.2}$.

Proof.

A result by Hamkins and Löwe [4, 3] shows that L has enough buttons and switches. Hence $\varphi \notin \mathsf{S4.2}$ implies $\varphi \notin \mathrm{MLF}(L)$. □

Proof of the Lemma.

Assign distinct buttons $b_{[v]}$ to all clusters $[v] \subseteq F$. Define

$$\begin{aligned} b_A &= \Big(\bigwedge_{[v]\in A} \Box b_{[v]}\Big) \wedge \Big(\bigwedge_{[v]\notin A} \neg\Box b_{[v]}\Big) && \text{for } A \subseteq F/{\equiv} \\ p_{[v]} &= \bigvee \{b_A \mid [v] = \bigvee A\} && \text{for } [v] \in F/{\equiv}. \end{aligned}$$

If $M[G] \models p_{[v]}$, then $M[G] \models \Diamond p_{[u]}$ iff $v \leq u$. W.l.o.g. $M \models p_{[w]}$.

Proof of the Lemma (continued).

Assume every cluster has at most 2^n nodes. For any cluster $[v] \in M/{\equiv}$, let $\{A_u : u \in [v]\}$ partition $\wp(n)$[2] for every cluster $[v] \subseteq F$ and define

$$s_A = \bigwedge_{i \in A} s_i \wedge \bigwedge_{i \notin A} \neg s_i \qquad \text{for } A \subseteq n$$

$$p_u = p_{[v]} \wedge \bigvee_{A \in A_u} s_a \qquad \text{for } u \in [v].$$

Then $\varphi_i = \bigvee \{p_u \mid (F, \leq, \|\cdot\|), u \models q_i\}$ works. □

[2] $\bigcup_{u \in [v]} A_u = \wp(n)$, $A_u \neq \emptyset$ and $A_u \cap A_{u'} = \emptyset$ for all $u \neq u' \in [v]$

Definition

The *maximality principle* MP is the scheme $\forall\varphi\,(\Diamond\Box\varphi \rightarrow \varphi)$.

Theorem (Hamkins [2])

ZFC and ZFC + MP are equiconsistent.

Theorem

If $M \models \mathsf{MP}$, then $\mathrm{MLF}(M) = \mathsf{S5}$.

Lemma

For any model M of set theory, $\{\varphi \mid M \models \Diamond\Box\varphi\}$ is consistent.

Proof.

Let $M \models \Diamond\Box\varphi_1, \ldots, \Diamond\Box\varphi_n$. Choose forcing posets such that $M^{\mathbb{P}_i} \models \Box\varphi_i$. Then $M^{\mathbb{P}_1 \times \cdots \times \mathbb{P}_n} \models \varphi_1, \ldots, \varphi_n$. □

Maximality Principle III

Theorem (Hamkins [2])

ZFC *and* ZFC + MP *are equiconsistent.*

Proof.

Let $M \models$ ZFC and $T = \{\varphi \mid M \models \Diamond\Box\varphi\}$. Since T is consistent, there is a model $N \models T \supseteq$ ZFC. Assuming $N \models \Diamond\Box\varphi$, we show $N \models \varphi$, i.e. $N \models$ MP.
If $M \models \Diamond\Box\neg\Diamond\Box\varphi$, then $\neg\Diamond\Box\varphi \in T$, so $N \models \neg\Diamond\Box\varphi$ in contradiction to $N \models \Diamond\Box\varphi$.
Hence $M \models \neg\Diamond\Box\neg\Diamond\Box\varphi \equiv \Box\Diamond\Diamond\Box\varphi$. By transitivity, $M \models \Box\Diamond\Box\varphi$. By reflexivity, $M \models \Diamond\Box\varphi$. Thus $\varphi \in T$ and $N \models \varphi$. $\Box$

Maximality Principle IV

Lemma

If $M \models \mathsf{MP}$, then so does every generic extension of M.

Proof.

Let N be a generic extension of M. Assume $N \models \Diamond\Box\varphi$. By transitivity, $M \models \Diamond\Box\Box\varphi$. By MP, $M \models \Box\varphi$. Hence $N \models \varphi$. □

Theorem

If $M \models \mathsf{MP}$, then $\mathrm{MLF}(M) = \mathsf{S5}$.

Proof.

By the above, $\Box(\Diamond\Box p \to p) \in \mathrm{MLF}(M)$. Since the forcing relation is reflexive and transitive, $\mathrm{MLF}(M) \supseteq \mathsf{S4} + \Diamond\Box p \to p = \mathsf{S5}$. $\mathrm{MLF}(M) \subseteq \mathsf{S5}$ holds for every M. □

Lemma

Let M' be a ground of M such that $M \models |\alpha| = \aleph_0$ for all ordinals α definable in M'. Then $\mathrm{MLF}(M) = \mathsf{S5}$.

Proof.

We show $M \models \mathsf{MP}$. Assume $M \models \Diamond\Box\varphi$. By transitivity, $M' \models \Diamond\Box\varphi$. In M', let

$$\alpha = \min\left\{2^{|\mathbb{P}|} \mid \mathbb{P} \text{ is a forcing with } \Vdash_{\mathbb{P}} \Box\varphi\right\}.$$

Then $M \models |\alpha| = \aleph_0$. Choose a forcing $\mathbb{P} \in M'$ such that $\Vdash_{\mathbb{P}} \Box\varphi$ in M' and $M \models (2^{|\mathbb{P}|})^{M'} = \aleph_0$. In M, $\mathbb{P}$ has only countably many dense subsets. Thus M can recursively define a $G \subseteq \mathbb{P}$ such that $N \models G$ is a $\mathbb{P}$-generic filter. Hence $M'[G] \subseteq M$. By Grigorieff's Theorem, $M'[G] \models \Box\varphi$ implies $M \models \varphi$. □

BBL

Definition

$(\mathrm{BBL}') = (\neg\Box p \wedge \Diamond\Box q) \to \Box(\Box p \to \Box q)$.
$\mathsf{BBL} = \mathsf{S4.2} + (\mathrm{BBL}')$.

Lemma

BBL *is sound and complete w.r.t. (finite) pre-Boolean algebras with at most two clusters. In particular,* $\mathsf{S4.2} \subsetneq \mathsf{BBL} \subsetneq \mathsf{S5}$.

Lemma

In a reflexive and transitive and directed Kripke model, $w \models (\mathrm{BBL}')$ *implies* $w \models \mathsf{BBL}$.

Proof.

By transitivity, $w \models (\mathrm{BBL}')$ implies $w \models \Box(\mathrm{BBL}')$. $\Box$

Definition

A sentence φ of set theory is a *final button* over M if it is an unpushed button over M and for any button ψ over M,

$$M \models \Diamond\left(\neg\Box\varphi \wedge \Box\psi\right) \rightarrow \Box\psi$$

as well as

$$M \models \Box\left(\Box\varphi \rightarrow \Box\psi\right).$$

Lemma

Let $M \models \mathsf{ZFC}$ with a final button φ over M and arbitrarily large finite families S of switches over M such that $S \cup \{\varphi\}$ is independent. Then $\mathrm{MLF}(M) = \mathsf{BBL}$.

Proof.

To show $\mathsf{BBL} \subseteq \mathrm{MLF}(M)$, it suffices to show that $(\mathrm{BBL}') \in \mathrm{MLF}(M)$.
Let ψ and χ be sentences of set theory.
We show $M \models (\neg\Box\psi \wedge \Diamond\Box\chi) \to \Box(\Box\psi \to \Box\chi)$.
If ψ is not a button over M, then $M \models \neg\Diamond\Box\psi \equiv \Box\neg\Box\psi$, so $M \models \Box(\Box\psi \to \Box\chi)$.
So assume ψ and χ are buttons over M and ψ is unpushed in M.
Since φ is a final button, $M \models \Diamond(\neg\Box\varphi \wedge \Box\psi) \to \Box\psi$ and $M \models \Box(\Box\varphi \to \Box\chi)$.
Since $M \models \neg\Box\psi$, we get $M \models \neg\Diamond(\neg\Box\varphi \wedge \Box\psi) \equiv \Box(\Box\psi \to \Box\varphi)$.
Hence $M \models \Box(\Box\psi \to \Box\chi)$.

Proof (continued).

To show $\mathrm{MLF}(M) \subseteq \mathsf{BBL}$, let $(F, \leq)$ be a pre-Boolean algebra with at most two clusters $[\bot] \leq [\top]$.
If $[\bot] = [\top]$, then $\mathrm{MLF}(M) \subseteq \mathsf{S5} \subseteq \mathrm{Log}(F, \leq)$.
Otherwise, use the unpushed final button to distinguish between the two clusters and the switches to distinguish between equivalent worlds. We obtain a labelling with sentences of set theory which shows $\mathrm{MLF}(M) \subseteq \mathrm{Log}(F, \leq)$. □

Theorem (Block and Hamkins)

There is a model of ZFC *whose modal logic of forcing is* BBL.

Proof.

Let $\boldsymbol{M}$ denote the mantle, i.e. the intersection of all grounds. Let $\boldsymbol{W}$ be a unary predicate symbol and define

$$S = \left\{\boldsymbol{W} \text{ is a ground } \wedge \omega_1 = \omega_1^{\boldsymbol{M}}\right\} \cup \left\{\Diamond\left(\omega_1 = \omega_1^{\boldsymbol{M}} \wedge \Box\varphi\right) \rightarrow \Box\varphi \mid \varphi \in \mathcal{L}_{\in,\boldsymbol{W}}\right\}.$$

Assume there is a model $(M, \in, M') \models \mathsf{ZFC} \cup S$. We claim that $\omega_1 \neq \omega_1^{\boldsymbol{M}}$ is a final button over M.

BBL as Modal Logic of Forcing II

Proof (continued).

For any $\mathcal{L}_{\in,\mathbf{W}}$-formula φ, $M \models \Diamond\left(\omega_1 = \omega_1^{\boldsymbol{M}} \wedge \Box\varphi\right) \to \Box\varphi$ implies

$$M \models \Diamond\left(\neg\Box\left(\omega_1 \neq \omega_1^{\boldsymbol{M}}\right) \wedge \Box\varphi\right) \to \Box\varphi$$

because $\omega_1 \neq \omega_1^{\boldsymbol{M}}$ is upwards absolute

Let $\alpha > \omega_1$ be definable in M'. Since there is a forcing collapsing α to ω_1 without collapsing ω_1 and $M \models \omega_1 = \omega_1^{\boldsymbol{M}}$,

$$M \models \Diamond\left(\neg\Box\left(\omega_1 \neq \omega_1^{\boldsymbol{M}}\right) \wedge \Box\left(|\alpha| = \omega_1^{\boldsymbol{M}}\right)\right)$$

Since α can be defined using an $\mathcal{L}_{\in,\mathbf{W}}$-formula, $M \models \Box\left(|\alpha| = \omega_1^{\boldsymbol{M}}\right)$.

Proof (continued).

Let N be a generic extension of M in which ω_1 is collapsed, i.e. $\omega_1^{\boldsymbol{M}} = \omega_1^M = \omega^N$.
Then $N \models |\alpha| = \omega_1^{\boldsymbol{M}}$, so

$$|\alpha|^N = \omega_1^{\boldsymbol{M}} = \omega_1^M = \omega^N.$$

Hence $\mathrm{MLF}(N) = \mathsf{S5}$. If φ is a button over M, then it also is over N, so $N \models \varphi$. Thus

$$M \models \Box\left(\Box\left(\omega_1 \neq \omega_1^{\boldsymbol{M}}\right) \to \Box\varphi\right)$$

for any button φ over M.
We conclude that $\left\{\omega_1 \neq \omega_1^{\boldsymbol{M}}\right\} \cup \left\{2^{\aleph_{\omega+n}} = \aleph_{\omega+n+1} \mid n \geq 1\right\}$ is an independent family of a final button and infinitely many switches over M. Hence $\mathrm{MLF}(M) = \mathsf{BBL}$.

Proof (continued).

It remains to show that $\mathsf{ZFC} \cup S$ is consistent. Let $\Phi = \{\varphi_0, \dots, \varphi_{n-1}\} \subseteq \mathcal{L}_{\in,\mathbf{W}}$ be finite. We show the consistency of

$$\mathsf{ZFC} \cup \left\{\boldsymbol{W} \text{ is a ground} \wedge \omega_1 = \omega_1^{\boldsymbol{M}}\right\} \cup \left\{\Diamond\left(\omega_1 = \omega_1^{\boldsymbol{M}} \wedge \Box\varphi\right) \to \Box\varphi \mid \varphi \in \Phi\right\}.$$

There is a model M' of ZFC with $M' \models \omega_1 = \omega_1^{\boldsymbol{M}}$. Let $M_0 = M'$.
If $(M_i, \in, M') \models \Diamond\left(\omega_1 = \omega_1^{\boldsymbol{M}} \wedge \Box\varphi_i\right)$, let M_{i+1} be a generic extension of M_i witnessing it. Otherwise, let $M_{i+1} = M_i$.
M' is a ground of M_n and $M_n \models \omega_1 = \omega_1^{\boldsymbol{M}}$. If $(M, \in, M') \models \Diamond\left(\omega_1 = \omega_1^{\boldsymbol{M}} \wedge \Box\varphi_i\right)$, then $(M_i, \in, M') \models \Diamond\left(\omega_1 = \omega_1^{\boldsymbol{M}} \wedge \Box\varphi_i\right)$ by transitivity. Hence $(M_{i+1}, \in, M') \models \Box\varphi_i$ and $(M_n, \in, M) \models \Box\varphi_i$. □

Corollary

There is a model M of ZFC *in whose multiverse* S4.2, BBL, *as well as* S5 *occur as modal logics of forcing.*

Proof.

A result by Piribauer [5] shows that S4.2 occurs in every multiverse.
The previous theorem shows that BBL can occur.
Its proof shows that S5 also occurs. □

Open questions

- Are any other modal logics possible?
- Are there multiverses with more than three modal logics of forcing?
- Is $\mathsf{MLF}(M)$ normal for every model M of ZFC?

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