Two days of Radin forcing

Recap from previous class:

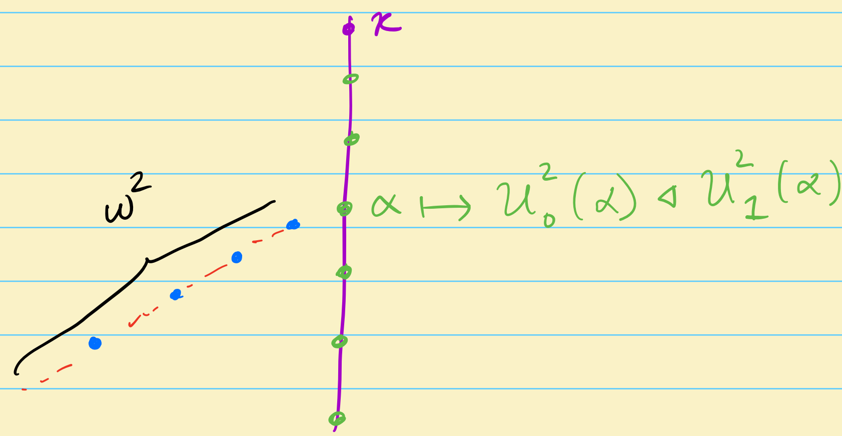
Radin forcing – Goals:

(1) Generalize Magidor's forcing.

(2) Shoot clubs consisting of former regulars while preserving large cardinals. $\square$

Motivating example (adding an ω^3-sequence)

Assuming $\mathcal{U}_0 \lhd \mathcal{U}_1 \lhd \mathcal{U}_2$, Magidor's forcing $\mathbb{M}(\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2)$ yields the following:



The basic building blocks of the Radin generic are measure sequences, which generalize $\triangleleft$-increasing seq of measures $\vec{u}$.

Prior to defining measure sequences we need the notion of a constructing embedding:

Definition:

Let $j: V \to M$ be an elementary embedding with $\mathrm{crit}(j) = \kappa$. Define

$$u^j = \langle u^j(\alpha) \mid \alpha < \ell(u^j) \rangle$$

as follows:

- $u^j(0) := \kappa$
- $u^j(\alpha) := \{ X \subseteq V_\kappa \mid u^j \upharpoonright \alpha \in j(X) \}$
- $\ell(u^j) := \min \{ \alpha \mid u^j \upharpoonright \alpha \notin M \}$.

Def: An elementary embedding $j: V \to M$ constructs u if there is $\alpha < \ell(u^j)$ s.t. $u = u^j \upharpoonright \alpha$.

Example:

- $u^j(1)$ is essentially the normal measure on κ derived from j.

$$u^j(1) \ni \{ \langle \alpha \rangle \mid \alpha < \kappa \wedge \alpha \text{ innacc} \}$$

- $u^j(2) := \{ X \subseteq V_\kappa \mid \langle \kappa, u^j(1) \rangle \in j(X) \}$

$$u^j(2) \ni \{ \langle \alpha, U \rangle \mid \alpha < \kappa \text{ meas} \wedge U \text{ meas on } V_\alpha \}$$

- $u^j(3) = \{ X \subseteq V_\kappa \mid \langle \kappa, u^j(1), u^j(2) \rangle \in j(X) \}$

$$u^j(3) \ni \{ \langle \alpha, U, V \rangle \mid \langle \alpha, U \rangle \text{ are as before}$$

$$\wedge\ V \ni \{ \langle \beta, W \rangle \mid \beta < \alpha \text{ meas} \wedge W \text{ meas on } V_\beta \} \}$$

- Forcing with $\mathbb{R}_{u^j \restriction 2}$ tantamounts to forcing with Prikry forcing. Forcing with $\mathbb{R}_{u^j \restriction 3}$ has the same effect as forcing with Magidor's forcing "$\mathbb{M}(u^j(1), u^j(2))$".

Definition (Measure sequences):

- $\mathcal{U}_0 = \{ u \mid \exists E \text{ extender } (j_E \text{ constructs } u) \}$
- $\mathcal{U}_{n+1} := \{ u \in \mathcal{U}_n \mid \forall \alpha \in (0, \ell(u)) \quad \mathcal{U}_n \cap V_{\kappa_u} \in u(\alpha) \}$

  $\kappa_u := u(0)$
- $\mathcal{U}_\infty = \bigcap_{n<\omega} \mathcal{U}_n$

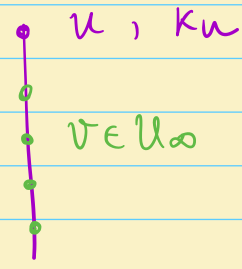
The collection of measure sequences is $\mathcal{U}_\infty$. $\square$

Note:

- Conditions in Rading forcing will mention members of $\mathcal{U}_\infty$ rather than $\mathcal{U}_0$
- The reason for this is securing the "fractal-like" architecture of Radin forcing. The key observation is:

  (*) If $u \in \mathcal{U}_\infty$ and $\alpha \in (0, \ell(u))$ then

  $$\mathcal{U}_\infty \cap V_{\kappa_u} \in u(\alpha).$$



Because of this observation it makes sense to define Radin for $\vec{v}$ etc.

Definition: Given $u \in \mathcal{U}_\infty$ we denote by $\mathcal{F}(u)$ the associated filter. That is,

$$\mathcal{F}(u) := \begin{cases} \{\phi\} & \text{if } \ell(u) = 1. \quad u = \langle \kappa_u \rangle. \\ \bigcap_{1 \le \alpha < \ell(u)} u(\alpha) & \text{if } \ell(u) \ge 2. \end{cases}$$

$\square$

Lemma (Woodin):

Suppose that $j : V \to M$ is an elementary embd. with $cp(j) = \kappa$ and $V_{\kappa+2} \subseteq M$. Then

(1) $\ell(u^j) \ge (2^\kappa)^+$

(2) $u^j \upharpoonright \alpha \in \mathcal{U}_\infty$ for all $\alpha < \ell(u^j)$. $\square$

Definition (Radin forcing):

Let $u \in \mathcal{U}_\infty$ with $\ell(u) \ge 2$.

The poset $\mathbb{R}_u$ has as conditions sequences

$$p = \langle (u_0, A_0), \dots, (u_i, A_i), \dots, (u, A_{\underbrace{n^p}_{\ell(p)}}) \rangle$$

stem of p.

where:

(1) For each $i < n^p$, $u_i \in \mathcal{U}_\infty$ and $(u_i, A_i) \in V_{\underbrace{\kappa_{u_{i+1}}}_{u_{i+1}(0)}}$

(2) For each $i < n^p$:

- $A_i \subseteq \{ v \in \mathcal{U}_\infty \mid \kappa_v > \kappa_{u_{i-1}} \}$
- $A_i \in \mathcal{F}(u_i)$.

Given $p, q \in \mathbb{R}_u$ we write $p \leq^* q$ if:

- $\ell(p) = \ell(q)$
- $u_i^p = u_i^q$ for all $i < \ell(p)$
- $A_i^p \subseteq A_i^q$ $\square$

Definition (One-point extension):

Given $p = \langle (u_0, A_0) \dots (u_i, A_i) \dots, (u, A_n) \rangle \in \mathbb{R}_u$ and $v \in A_i$ we define $p^\frown \langle v \rangle$ as the sequence

$$p^\frown \langle v \rangle := \langle (u_0, A_0), \dots, (u_{i-1}, A_{i-1}) \, (v, A_i \cap V_{\kappa_v}), (u_i, \{ w \in A_i \mid \kappa_w > \kappa_v \})$$

$(u_{i+1}, A_{i+1}) \ldots, (u, A_n) \rangle$.

More generally, given $\vec{v} \in \mathcal{U}_\infty^{<\omega}$ we define $p \curvearrowright \vec{v}$ by recursion in the obvious fashion. $\square$

Def: A condition $p \in \mathbb{R}_u$ is <u>pruned</u> if for all $\vec{v} \in \mathcal{U}_\infty^{<\omega}$, $p \curvearrowright \vec{v} \in \mathbb{R}_u$.

Lemma: $\{p \in \mathbb{R}_u \mid p \text{ is pruned}\}$ is $\leq^*$-dense. $\square$

Definition: Given $p, q \in \mathbb{R}_u$ we write $p \leq q$ if there is $\vec{v} \in \mathcal{U}_\infty^{<\omega}$ s.t $p \leq^* q \curvearrowright \vec{v}$.

Example: Suppose that $u \in \mathcal{U}_\infty$ has $\ell(u) = 2$. This means that $u = \langle \kappa, U \rangle$

Clearly, $\underbrace{\langle (u, \{\langle \alpha \rangle \mid \alpha < \kappa\}) \rangle}_{p} \in \mathbb{R}_u$.

$F(u) = u(1) = U$.

If we force with $\mathbb{R}_u / p$ we know that every $q \in \mathbb{R}_u/p$ is of the form

$$q \le^* p \frown \vec{v}, \text{ for some } \vec{v} \in U_\infty^{<\omega}.$$

For instance, suppose $q \le^* p \frown \langle v \rangle$

- $v = \langle \alpha \rangle$
- $p \frown \langle v \rangle = \langle (\underbrace{\langle \alpha \rangle}_{v}, \emptyset), (u, \{ \langle \beta \rangle \mid \beta > \alpha \}) \rangle$
- $q \le^* p \frown \langle v \rangle$ is of the form

$$q = \langle (\langle \alpha \rangle, \emptyset), (u, A) \rangle.$$

In general,

$$q = \langle (\langle \alpha_0 \rangle, \emptyset), (\langle \alpha_1 \rangle, \emptyset), \dots, (\langle \alpha_{n-1} \rangle, \emptyset), (u, A) \rangle$$

This is Prikry forcing in disguise. $\square$

The next lemma makes precise the "fractal-like" architecture of $\mathbb{R}_u$:

Lemma: Let $p \in \mathbb{R}_u$ and $i < \ell(p)$. Then

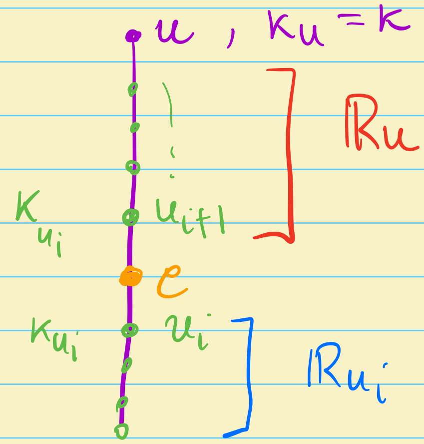
$$\mathbb{R}_u/p \cong \mathbb{R}_{u_i}/p\restriction i+1 \times \mathbb{R}_u/\langle(\kappa_{u_i}, \phi)\rangle^\frown p\setminus i+1$$

$$\left(\begin{array}{l}\text{where:} \\ \quad p\restriction i+1 = \langle (u_0, A_0), \ldots, (u_i A_i)\rangle \\ \quad p\setminus i+1 = \langle (u_{i+1}, A_{i+1}), \ldots, (u, A_n)\rangle \end{array}\right).$$

as witnessed by the map:

$$q \leq p \longmapsto \langle q\restriction i+1, \langle(\kappa_{u_{i+1}}, \phi)\rangle^\frown q\setminus i+1\rangle.$$

$\square$



Lemma: $\mathbb{R}_u$ has the Prikry property. Namely, for each $p \in \mathbb{R}_u$ and φ a sentence in the language of forcing there is $q \leq^* p$ s.t $q \Vdash \varphi$ or $q \Vdash \neg \varphi$. $\boxtimes$

The generic Radin object:

Define: Let $G \subseteq \mathbb{R}_u$ be generic. Define:

$$MS_G = \{ v \in \mathcal{U}_\infty \mid \exists p \in G \ (v \text{ occurs in } p) \}.$$

The Radin sequence inferred from G is

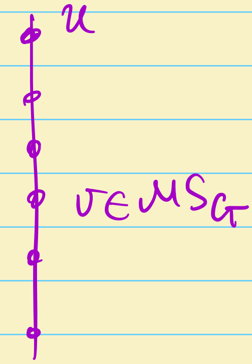
$$\langle v_\alpha \mid \alpha < \chi_G \rangle$$

the increasing $(v < w \iff \kappa_v < \kappa_w)$ enumeration of MS_G.

The Radin club inferred from G is

$$C(G) = \langle \kappa_{v_\alpha} \mid \alpha < \chi_G \rangle$$ $\square$

Remark: Given any $\alpha \in C(G)$ there is a <u>unique</u> $v \in \mathcal{MS}_G$ s.t. $\alpha = \kappa_v$.



Theorem: $V[G] = V[\mathcal{MS}_G]$. ($= V[C(G)]$ in some situation)

Lemma: Let $G \subseteq \mathbb{R}_\mathcal{U}$ be generic. Then

$$C(G) = \langle \kappa_{v_\alpha} \mid \alpha < \chi_G \rangle$$

is a club subset of $\kappa_\mathcal{U}$.

Proof: Set $C := C(G)$ and $\kappa := \kappa_\mathcal{U}$

Unbounded: Let $\alpha < \kappa$ and look at

$$D_\alpha = \{ p \in \mathbb{R}_\mathcal{U} \mid \exists v \in \mathcal{U}_\infty \, (v \text{ occurs in } p \wedge \kappa_v > \alpha) \}.$$

If D_α is dense then $D_\alpha \cap G \ni p$ and the there is $v \in \mathcal{MS}_G$ s.t. $\kappa_v > \alpha$.

D_α is dense: Let $p \in \mathbb{R}_u$

$$p = \langle (u_0, A_0), \dots, (u_i, A_i) \dots (u, A) \rangle$$

where $i \leq \ell(p)$ is the first index with $\alpha < \kappa_{u_i}$.

Define

$$B = \{ v \in A_i \mid \kappa_v > \alpha \} \in \mathcal{F}(u_i)$$

and look at $p^* = \langle (u_0, A_0) \dots, (u_i, B), \dots \rangle$

Then $p^* \frown \langle v \rangle = \langle (u_0, A_0), \dots, (v, B \cap V_{\kappa_v})$

$, (u_i, B^*) \dots \rangle$ and $p^* \frown \langle v \rangle \in D_\alpha$.

Clubness: Suppose that $\alpha < \kappa$ s.t $\alpha \notin C$.

We show that $C \cap \alpha$ is bounded in α.

Let $p \in G$, $p \Vdash$ "$\alpha \notin \dot{C}$" and $i \leq \ell(p)$

$\kappa_{u_i^p} \leq \alpha < \kappa_{u_{i+1}^p}$. Note that

$\kappa_{u_i^p} < \alpha < \kappa_{u_{i+1}^p}$ (because $p \Vdash \alpha \notin \dot{C}$).

Then, $p = s ^\frown (\langle \kappa^p_{u_{i+1}} \rangle, \phi) ^\frown s_1$.

Therefore $\underset{\substack{\cap \\ G}}{p} \Vdash \sup(\dot{C} \cap \alpha) = \kappa^p_{u_i} < \alpha$. $\blacksquare$