Two days of Radin forcing

Recap from previous class:

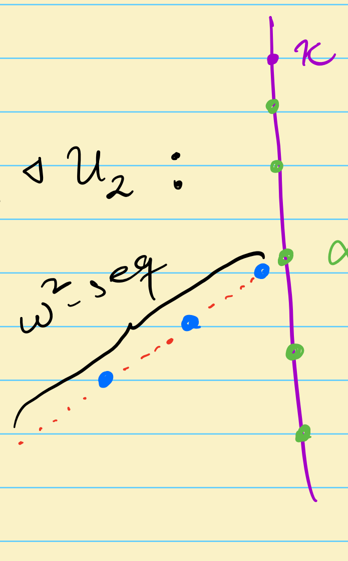
Radin forcing - Goals:

(1) Generalize Magidor's forcing

(2) Shoot clubs consisting of former regulars while preserving large cardinals. ∎

Motivating example (adding ω^3-sequence)

Assuming $\mathcal{U}_0 \lhd \mathcal{U}_1 \lhd \mathcal{U}_2$:



$$\alpha \mapsto \mathcal{U}_0^2(\alpha) \lhd \mathcal{U}_1^2(\alpha)$$

$$\rightsquigarrow \mathbb{M}(\mathcal{U}_0^2(\alpha), \mathcal{U}_1^2(\alpha))$$

∎

The basic building blocks of Radin forcing are measure sequences, which aim to generalize $\triangleleft$-increasing seq. of measures.

Prior to defining measure sequences we need the notion of constructing embedding:

Definition:

Let $j : V \longrightarrow M$ be an elementary embedding with $\mathrm{crit}(j) = \kappa$. Define

$$u^j = \langle u^j(\alpha) \mid \alpha < \ell(u^j) \rangle$$

where:

- $u^j(0) := \kappa$.
- $u^j(\alpha) := \{ X \subseteq V_\kappa \mid u^j \restriction \alpha \in j(X) \}$
- $\ell(u^j) := \min \{ \alpha \mid u^j \restriction \alpha \notin M \}$.

Def An elementary embedding j constructs u if there is $\alpha < \ell(u^j)$ s.t. $u = u^j \restriction \alpha$. $\square$

Example:

- $u^j(1)$ is essentially the normal measure on κ derived from j.

$$u^j(1) \ni \{ \langle \alpha \rangle \mid \alpha \text{ inacc } \wedge \alpha < \kappa \}$$

- $u^j(2) = \{ X \subseteq V_\kappa \mid \langle \kappa, u^j(1) \rangle \in j(X) \}$

$$u^j(2) \ni \underbrace{\{ \langle \alpha, U \rangle \mid \alpha \text{ is meas } \wedge U \text{ is } \alpha\text{-complete uf on } \{ \langle \beta \rangle \mid \beta < \alpha \} \}}_{X_\kappa}$$

- $u^j(3) = \{ X \subseteq V_\kappa \mid \langle \kappa, u^j(1), u^j(2) \rangle \in j(X) \}$

$$u^j(3) \ni \{ \langle \alpha, U, V \rangle \mid \langle \alpha, U \rangle \in X_\kappa \wedge V \ni \{ \langle \beta, W \rangle \mid \beta < \alpha \wedge W \text{ meas on } V_\alpha \} \}$$

- Forcing with $\mathbb{R}_{u^j \restriction 2}$ is like forcing with Prikry forcing (i.e., Radin forcing of order 1) while forcing with $\mathbb{R}_{u^j \restriction 3}$ is like forcing with Magidor forcing "$\mathbb{M}(u^j(1), u^j(2))$".

Definition (Measure sequences):

- $\mathcal{U}_0 = \{ u \mid \exists E \text{ extender } (j_E \text{ constructs } u) \}$.
- $\mathcal{U}_{n+1} = \{ u \in \mathcal{U}_n \mid \forall \alpha \in (0, \ell(u)) \quad \mathcal{U}_n \cap V_{\kappa_u} \in u(\alpha) \}$
- $\mathcal{U}_\infty = \bigcap_{n < \omega} \mathcal{U}_n$

The collection of measure sequences is $\mathcal{U}_\infty$. $\square$

(we'll denote $\kappa_u := u(0)$)

Note:

- Conditions in Radin forcing will mention members of $\mathcal{U}_\infty$ rather than $\mathcal{U}_0$ (as it might have been originally expected).

- The reason for this is securing the "fractal-like" architecture of Radin forcing. The key observation being the following:

(*) If $u \in \mathcal{U}_\infty$ has $\ell(u) \geq 2$ then

$$\forall \alpha \in (0, \ell(u)), \quad \mathcal{U}_\infty \cap V_{\kappa_u} \in u(\alpha).$$

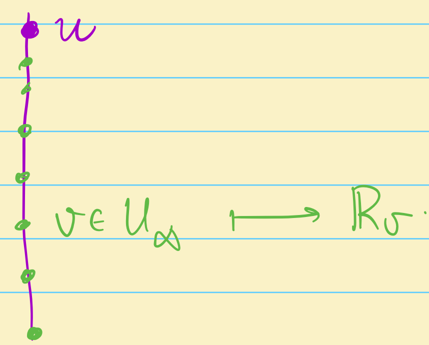
(i.e., $u(\alpha)$ concentrates on measure seq, therefore on objects of the same type!)

Because of the above observation it makes sense to define the Radin forcing for $v \in X \in u(\alpha)$ $\square$

Definition: Given $u \in \mathcal{U}_\infty$ we denote by $\mathcal{F}(u)$ the associated filter – that is

$$\mathcal{F}(u) := \begin{cases} \{\emptyset\} & \text{if } \ell(u) = 1 \\ \bigcap_{\alpha \in (0, \ell(u))} u(\alpha) & \text{if } \ell(u) \geq 2. \end{cases}$$ $\square$

Following the intuition with Magidor's forcing we'll have



The next lemma shows that long enough measure sequences exist under appropriate large cardinal hypothesis:

Lemma (Woodin):

Suppose that $j : V \to M$ is an elementary embedding with $\mathrm{crit}(j) = \kappa$ and $M \supseteq V_{\kappa+2}$. Then:

(1) $\ell(u^j) \geq (2^\kappa)^+$.

(2) $u^j \upharpoonright \alpha \in \mathcal{U}_\infty$ for all $\alpha < (2^\kappa)^+$. ∎

Definition (Radin forcing):

Let $u \in \mathcal{U}_\infty$ be with $\ell(u) \geq 2$.

The poset $\mathbb{R}_u$ has as conditions sequences

$$p = \langle (u_0, A_0), \ldots, (u_i, A_i), \ldots (u, \underbrace{A_n}_{e(p)}) \rangle$$

where:

(1) For each $i < n$, $u_i \in \mathcal{U}_\infty$ and $(u_i, A_i) \in V_{\kappa_{u_{i+1}}}$

(2) For each $i < n$:

- $\{v \in \mathcal{U}_\infty \mid \kappa_v > \kappa_{u_{i-1}}\} \supseteq A_i$
- $A_i \in \mathcal{F}(u_i)$.

Given $p, q \in \mathbb{R}_u$ we write $p \leq^* q$ if

- $\ell(p) = \ell(q)$
- $u_i^p = u_i^q$ for all $i < \ell(p)$.
- $A_i^p \subseteq A_i^q$ for all $i < \ell(p)$. ∎

In order to define the main order of $\mathbb{R}_u$ we first need to introduce the notion of a one-point-extension:

Definition. Given a condition

$$p = \langle (u_0, A_0), \ldots, (u_i, A_i), \ldots, (u, A_n) \rangle$$

and $v \in A_i$, we define $p \frown \langle v \rangle$ as

$$p \frown \langle v \rangle = \langle (u_0, A_0), \ldots, (v, A_i \cap V_{\kappa_v}), (u_i, \{w \in A_i \mid \kappa_v < \kappa_w\}), \ldots, (u, A_n) \rangle$$

More generally, given $\vec{v} \in \mathcal{U}_\infty^{<\omega}$ we define

$$p \frown \vec{v}$$

by recursion in the obvious fashion. ⊠

Note that, in principle, nothing prevents for $p^\frown\langle v\rangle$ not to be a condition. Fortunately, there is a $\leq^*$-dense set of conditions for which this is the case:

Def: A condition $p \in \mathbb{R}_u$ is called pruned if $p^\frown \vec{v} \in \mathbb{R}_u$ for all $v \in \mathcal{U}_\infty^{<\omega}$.

Lemma: $\{p \in \mathbb{R}_u \mid p \text{ is pruned}\}$ is $\leq^*$-dense in $\mathbb{R}_u$.

Proof: Let $p = \langle (u_0, A_0), \dots, (u_i, A_i), \dots, (u, A_n)\rangle$

For each $i \leq n$ define:

- $B_i^0 := A_i$
- $B_i^{(n+1)} := \{v \in B_i^{(n)} \mid B_i^{(n)} \cap V_{\kappa_v} \in \mathcal{F}(v)\}$
- $B_i := \bigcap_{n<\omega} B_i^{(n)}$

Claim: $B_i^{(n)} \in \mathcal{F}(u_i)$. In particular, $B_i \in \mathcal{F}(u_i)$

Proof: This is proved by induction on $n < \omega$.

Suppose $B_i^{(n)} \in \mathcal{F}(u_i)$. Then $B_i^{(n+1)} \in \mathcal{F}(u_i)$ because, for each $\alpha < \ell(u_i)$,

$$B_i^{(n+1)} \in u_i(\alpha) \Longleftrightarrow u_i \upharpoonright \alpha \in j(B_i^{(n+1)})$$

$$\Longleftrightarrow j(B_i^{(n)}) \cap V_{\underbrace{\kappa_{u_i \upharpoonright \alpha}}_{\kappa}} = B_i^{(n)} \in \mathcal{F}(u_i \upharpoonright \alpha)$$

($\checkmark$)

The last assertion in the claim follows from σ-completeness. ∎

Let $p^* = \langle (u_0, B_0), \ldots, (u_i, B_i), \ldots, (u, B_n) \rangle$.

Clearly, $p^* \leq^* p$ and it's easy to check that p^* is pruned:

Suppose that $p^* \frown \vec{v} \in \mathbb{R}_u$. Now let w in one of the measure one sets of $p^* \frown \vec{v}$. We claim that $p^* \frown \vec{v} \frown \langle w \rangle \in \mathbb{R}_u$.

Indeed, if $w \in B_i$ then $B_i \cap V_{\kappa_w} \in \mathcal{F}(w)$ and so $p^* \frown \vec{v} \frown \langle w \rangle \in \mathbb{R}_u$. Otherwise $w \in B_i \cap V_{\kappa_{v_t}}$. In that case

$$B_i \cap V_{\kappa_w} = \bigcap_{n<\omega} (B_i^n \cap V_{\kappa_w}) \in \mathcal{F}(w)$$

because $w \in B_i = \bigcap_{n<\omega} B_i^n$. $\boxtimes$

Definition: Given $p, q \in \mathbb{R}_u$ write $p \leq q$ if there is $\vec{v} \in \mathcal{U}_\infty^{<\omega}$ s.t. $p \leq^* q \frown \vec{v}$. $\square$

Exercise: Check that $\leq$ is transitive.

Illustrating Radin forcing: Let $u \in \mathcal{U}_\infty$ be with $\ell(u) \geq 2$.

$\ell(u) = 2$: Then $u = \langle \kappa, U \rangle$.

Clearly, $\langle u, \{\langle \alpha \rangle \mid \alpha < \kappa\} \rangle \in \mathbb{R}_u$. Taking this as the trivial condition (i.e., forcing with $\mathbb{R}_u / \langle u, \{\langle \alpha \rangle \mid \alpha < \kappa\} \rangle$)

we know that every condition is a $\leq^*$-ext of

$$\langle u, \{\langle\alpha\rangle \mid \alpha<\kappa\}\rangle ^\frown \vec{v} \quad , \text{ for } \vec{v} \in \mathcal{U}_\infty^{<\omega}$$

Specifically, conditions are of the form

$$\langle \langle \langle\alpha_0\rangle, \phi\rangle, \ldots, \langle\langle\alpha_{n-1}\rangle, \phi\rangle, \langle u, \underset{\{\langle\alpha\rangle \mid \alpha>\alpha_{n-1}\}}{\underset{\cap\mathsf{I}}{B}}\rangle\rangle$$

This is just Prikry forcing in disguise.

$\ell(u)=3$: $\quad u = \langle \kappa, U_0, U_1\rangle$

Let $X_0 = \{\langle\alpha\rangle \mid \alpha<\kappa\}$, $X_1 = \{\langle\alpha, U\rangle \mid \alpha$ meas$\}$ U meas on V_α

and $p_0 = \langle u, X_0 \cup X_1\rangle$.

Then $\mathbb{R}u/p_0$ is isomorphic to Magidor's forcing

Conditions are vectors

$$p = \langle \ldots, \langle\langle\alpha_0\rangle, \phi\rangle, \ldots, \langle\langle\alpha_i\rangle, \phi\rangle \langle\langle\alpha_{i+1}, W_{i+1}\rangle A_{i+1}\rangle$$
$$, \ldots, \langle\langle\alpha_j, W_j\rangle, A_j\rangle, \ldots, (u, \underset{X_0 \cup X_1}{\underset{\cap\mathsf{I}}{B}})\rangle$$

$\kappa_u \leq \ell(u) \leq \kappa_u^+$

- If $\ell(u) = \kappa_u$, then $\mathbb{R}_u$ forces "$\mathrm{cof}(\kappa_u) = \omega$"!!
- If $\ell(u) = \kappa_u^+$ then $\mathbb{R}_u$ preserves the regularity of κ_u.
- If $\ell(u) \in (\kappa_u, \kappa_u^+)$, then $\mathbb{R}_u$ forces "$\mathrm{cof}(\kappa_u) = \mathrm{cof}(\ell(u))$".

Basic properties of $\mathbb{R}_u$:

Lemma: Let $u \in \mathcal{U}_\infty$ be with $\ell(u) \geq 2$. Then $\mathbb{R}_u$ is κ_u^+-Knaster.

Proof: If two conditions p, q have the same stem then they're $\leq^*$-compatible. There are at most κ_u-many stems, so $\mathbb{R}_u$ is κ_u^+-Knaster. $\square$

The next lemma makes precise our idea around the fractal-like architecture of $\mathbb{R}_u$:

Lemma Let $p \in \mathbb{R}_u$ and $i < \ell(p)$. Then

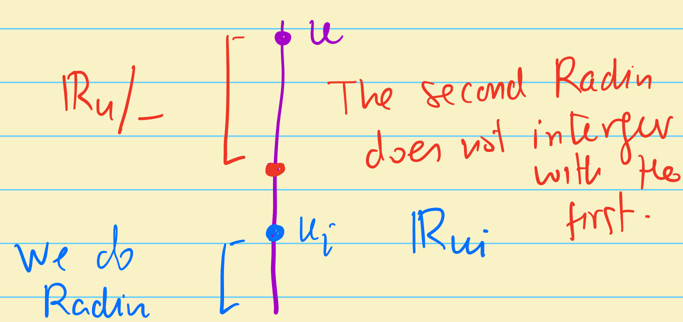
$$\mathbb{R}_u/p \cong \mathbb{R}_{u_i}/p \restriction i+1 \quad \times \quad \mathbb{R}_u/\langle\langle \kappa_{u_i} \rangle, \phi\rangle ^\frown p \setminus i+1$$

as witnessed by the map.

$$q \mapsto \langle q \restriction i+1 \,,\ \langle\langle \kappa_{u_i}\rangle, \phi\rangle ^\frown q \setminus i+1 \rangle.$$

Note:

- Observe that the first of this factors is a miniaturized Radin forcing while the other is a Radin forcing based on u whose $\leq^*$-order is $\kappa_{u_i}^+$-closed



Lemma $\mathbb{R}_u$ has the Prikry property. That is, given any $p \in \mathbb{R}_u$ and any sentence φ in the language of forcing of $\mathbb{R}_u$, there is $q \leq^* p$ s.t either $q \Vdash \varphi$ or $q \Vdash \neg \varphi$.

Proof idea: Let $p = \langle (u, A) \rangle \in \mathbb{R}_u$.

Step 1: For each $s \in [\bar{V}_{\kappa_u}]^{<\omega}$ that serves as a stem split A into three sets:

$$A^0_s := \{ v \in A_0 \mid \exists B_1, B_2 (s ^\frown \langle v, B_1 \rangle ^\frown \langle u, B_2 \rangle \Vdash \varphi) \}$$

$$A^1_s := \{ v \in A_0 \mid \exists B_1 B_2 (s ^\frown \langle v, B_1 \rangle ^\frown \langle u, B_2 \rangle \Vdash \neg \varphi) \}$$

$$A^2_s := A \setminus (A^0_s \cup A^1_s).$$

For each $\alpha \in (0, \ell(u))$, $A \in u(\alpha)$, so that a unique $A^i_s \in u(\alpha)$. Let $i_\alpha \in \{0, 1, 2\}$ be the unique index s.t $[A^{i_\alpha}_s \in u(\alpha).]$

Now we get rid of the dependence on s taking "diagonal intersections". That is, define

$$A(\alpha) := \{ v \in A \mid \forall s \in [V_{\kappa_u}]^{<\omega} \ (s \in V_{\kappa_v} \rightarrow v \in A_s^{i_\alpha}) \}.$$

One can show that $A(\alpha) \in \mathcal{U}(\alpha)$.

In the end we let $A^* = \bigcup_{\alpha \in (0, \ell(u))} A(\alpha)$ which is $\mathcal{F}(u)$-large.

Step 2: In Step 1 we defined a condition

$$p^* = \langle (u, A^*) \rangle \leq^* p$$

The property of p^* is the following.

Let $q \leq p^*$ be a condition of the form

$$\underbrace{s_0 \frown \langle (v, X) \rangle}_{s} \frown \langle (u, B) \rangle$$

that decides φ and s.t. $|s|$ is minimal

(Let's say that $q \Vdash \varphi$).

By definition, $q \leq^* p ^\frown \vec{w} ^\frown \langle v \rangle$, so that $v \in A^* = \bigcup_\alpha A(\alpha)$. Let α be s.t $v \in A(\alpha)$.

By the def of $A(\alpha)$, $v \in A^{i_\alpha}_{s_0}$.

Also, since $q \Vdash \varphi$ it must be that $i_\alpha = 0$

Therefore, for all $w \in A(\alpha)$ with $s_0 \in V_{\kappa_w}$ there are sets B^0_w, B^1_w s.t

$$s_0 ^\frown \langle w, B^0_w \rangle ^\frown \langle u, B^1_w \rangle \Vdash \varphi.$$

Step 3: We integrate all of this B^0_w's, B^1_w.

This goes as follows:

- $B^{up} := \underbrace{\{ z \in A^* \mid \forall w \, (w \in V_{\kappa_z} \rightarrow z \in B^1_w)\}}_{\text{Diagonal intersection of } B^1_w\text{'s}}$

- B^{bottom} is defined as the union of three sets.

$$B^{<\alpha} = j(w \mapsto B^0_w)(u\restriction\alpha) \in \mathcal{F}(u\restriction\alpha)$$

$$B^{\alpha} = \{ w \in A^* \mid B^{<\alpha} \cap V_{\kappa_w} \in \mathcal{F}(w) \} \in u(\alpha)$$

$$B^{>\alpha} = \{ w \in A^* \mid \exists \xi < \ell(w) \quad B^{\alpha} \cap V_{\kappa_w} \in w(\xi) \} \underset{\alpha<\beta<\ell(u)}{\in \bigcap u(\beta)}$$

Therefore $B^{\text{bottom}} = B^{<\alpha} \cup B^{\alpha} \cup B^{\geq \alpha}$.

Define $p^{**} = \langle u, B^{\text{top}} \cap B^{\text{bottom}} \rangle \leq^* p^*$

Step 4: One now shows that

$$q^* = s_0 \frown \langle u, B^{\text{top}} \cap B^{\text{bottom}} \rangle$$

is an extension of p^*, with length $< \ell(q)$ that decides φ. Contradiction!! $\square$

The generic Radin object:

Def: Given $p \in \mathbb{R}_u$ and $v \in \mathcal{U}_\infty$ we say that v occurs in p if there is $i < \ell(p)$ s.t $u_i^p = v$. ∎

Definition: Let $G \subseteq \mathbb{R}_u$ be generic. Write:

$$MS_G = \{ v \in \mathcal{U}_\infty \mid \exists p \in G \, (v \text{ occurs in } p) \}$$

The Radin sequence inferred from G is

$$\langle v_\alpha \mid \alpha < \chi_G \rangle$$

, the increasing enumeration of MS_G.

The Radin club inferred from G is

$$\langle \kappa_{v_\alpha} \mid \alpha < \chi_G \rangle .$$

Remark: For each $\alpha \in C$ there is at most one $v \in MS_G$ s.t $\alpha = \kappa_v$. ∎

Lemma: Let $G \subseteq \mathbb{R}_u$ be generic. Then $V[MS_G] = V[G]$.

Proof: One can show that

$$G = \{ p \in \mathbb{R}_u \mid \forall v \, (v \text{ occurs in } p \Rightarrow v \in MS_G)$$
$$\wedge \, \forall v \in MS_G \, \exists q \leq p \, (v \text{ occurs in } q) \} . \qquad \boxtimes$$

Lemma: Let $G \subseteq \mathbb{R}_u$ be generic. Then

$$C(G) = \langle \kappa_{v_\alpha} \mid \alpha < \chi_G \rangle$$

is a club subset of κ.

Proof: Set $C := C(G)$

Unbounded: Let $\alpha < \kappa$. Then look at

$$D_\alpha = \{ p \in \mathbb{R}_u \mid \exists v \in \mathcal{U}_\infty \, (v \text{ occurs in } p \text{ and } \kappa_v > \alpha) \}$$

We claim thet this is dense.

Indeed, let $p \in \mathbb{R}_u$. Let $i \leq \ell(p)$ be the first index s.t. $\alpha < \kappa_{u_i}$. Since p is pruned and

$$\{ v \in A_i \mid \kappa_v > \alpha \} \in \mathcal{F}(u_i)$$

we have that $p ^\frown \langle v \rangle \leq p$ has the required property.

Closed: Suppose that $\alpha < \kappa$ is s.t. $\alpha \notin C$. We show that $C \cap \alpha$ is bounded in α. Let $p \in G$ s.t. $p \Vdash \alpha \notin C$. Let $i < \ell(p)$ be the first s.t. $\kappa_{u_i^p} \leq \alpha < \kappa_{u_{i+1}^p}$. Note that, in fact, $\kappa_{u_i^p} < \alpha < \kappa_{u_{i+1}^p}$. Now note that

$$p = s \frown \langle \langle \kappa_{u_{i+1}^p} \rangle, \phi \rangle \frown s_1$$

Therefore, $p \Vdash$ "$\sup(\dot{C} \cap \alpha) = \kappa_{u_i^p} < \alpha$"

Since $p \in G$, $\sup(C \cap \alpha) < \alpha$, as needed. ∎

Lemma: Let $u \in \mathcal{U}_\infty$ be with $2 \leq \ell(u)$.

Then, $\Vdash_{\mathbb{R}_u}$ "$\mathrm{otp}(C(\dot{G})) = \omega^{\ell(u)}$".

In particular, if $\mathrm{cof}^V(\ell(u)) \geq \omega_1$,

$\langle (u, \{v \in \mathcal{U}_\infty \mid \kappa_v > \mathrm{cof}(\ell(u))\}) \rangle \Vdash_{\mathbb{R}_u}$ "$\mathrm{cof}(\kappa) = \mathrm{cof}^V(\ell(u))$". ∎

Theorem (Cardinal structure):

Let $u \in \mathcal{U}_\infty$ be with $\ell(u) \geq 2$ and $G \subseteq \mathbb{R}_u$ generic. Denote $\langle \kappa_\alpha \mid \alpha < \chi_G \rangle$ the enumeration of the Radin club. Then:

(1) For each ordinal $e < \kappa$,

$$\mathcal{P}(e)^{V[G]} = \mathcal{P}(e)^{V[G \restriction \alpha]}$$

where α is the first index $< \chi_G$ s.t

$$\kappa_\alpha \leq e < \kappa_{\alpha+1}$$

and $G \restriction \alpha$ is the generic induced by $\mathbb{R}_u/p \to \mathbb{R}_{u_\alpha}/p$ for u_α the α^{th}-member of MS_G.

(2) $\mathbb{R}_u$ preserves cardinals.

Proof:

(1) Fix $e < \kappa$. Clearly, it suffices to show

$$\mathcal{P}(e)^{V[G]} \subseteq \mathcal{P}(e)^{V[G \restriction \alpha]}$$

Suppose that $a \in \mathcal{P}(e)^{V[G]}$ and let $\alpha < \chi_G$ as in the statement of the theorem.

Let $p \in G$ be s.t $u_\alpha, u_{\alpha+1}$ (i.e the α and $(\alpha+1)^{th}$-members of $\mathcal{MS}_G$) occur in p.

Then,

$$\mathbb{R}_u / p \cong \overbrace{\mathbb{R}_{u_\alpha} / p \restriction i{+}1}^{\mathbb{R}_1} \times \overbrace{\mathbb{R}_u / \langle (\kappa_{\alpha+1}, \phi) \rangle ^\frown p \setminus i{+}1}^{\mathbb{R}_2}$$

Note that $\langle \mathbb{R}_2, \leq^* \rangle$ is $\kappa_{\alpha+1}^+$ - closed.

For each $\sigma < e$ let $\varphi_\sigma \equiv$ "$\sigma \in \dot{a}_{G\restriction\alpha}$"

where $\dot{a}_{G\restriction\alpha}$ is an $\mathbb{R}_2$-name s.t $\dot{a}_{G\restriction\alpha \times G\setminus\alpha} = a$.

Combining the Prikry property of $\mathbb{R}_2$ with the e^+-closure of $\leq^*$ we find

$$V[G\restriction\alpha] \ni \langle p_1^\sigma \mid \sigma \leq e \rangle \quad \leq^*\text{-decreasing}$$

s.t $V[G\restriction\alpha] \models$ "$p_1^\sigma \Vdash \sigma \in \dot{a}_{G\restriction\alpha}$" for $\sigma < e$.

By density, we can assume that $p_1^e \in G\setminus\alpha$

Define

$$b = \{ \sigma < e \mid p_1^e \Vdash^{V[G\restriction\alpha]} \check{\sigma} \in \dot{a}_{G\restriction\alpha} \}.$$

Note that $p_1^e \Vdash^{V[G\restriction\alpha]} \check{b} = \dot{a}_{G\restriction\alpha}$.

Therefore $b = a$ and so $a \in \mathcal{P}(e)^{V[G\restriction\alpha]}$. $\square$

(2) Since $\mathbb{R}_u$ is κ^+-Knaster, cardinals $\geq \kappa^+$ are preserved. The preservation of κ will follow from all cardinals $e < \kappa$ being preserved. Let's argue this.

Suppose that $e < \kappa$ is a V-cardinal that ceases to be so in $V[G]$. Let $f \in V[G]$

$$f : \sigma \longrightarrow e$$

be a surjection, for some $\sigma < e$.

Thus (in $V[G]$) there is $a \subseteq \sigma$ with order-type $\geq e$. We show that this is impossible. By the previous argument we know that

$$a \in \mathcal{P}(\sigma)^{V[G\restriction\alpha]}$$

where α is the first ordinal s.t. $\kappa_\alpha \leq \sigma < \kappa_{\alpha+1}$.

That means that in $V[G \restriction \alpha] \models |e| \leq |\sigma|$.

However, this is a generic extension by a $\kappa_\alpha^+ (\leq e)$- Knaster forcing, so

$$V[G \restriction \alpha] \models |\sigma| \leq \sigma < e = |e|$$

This is a contradiction. $\boxtimes$

Some additional results:

The cofinality of κ when $\ell(u) \in \{\kappa_u, \kappa_u^+\}$:

Theorem (Gitik? Mitchell?)

(1) If $\ell(u) = \kappa_u$ then $\Vdash_{\mathbb{R}_u}$ "$\mathrm{cof}(\kappa_u) = \check{\omega}$".

(2) If $\ell(u) = \kappa_u^+$ then $\Vdash_{\mathbb{R}_u}$ "$\mathrm{cof}(\kappa_u) = \kappa_u$".

Proof: We just prove (1). (2) needs the Strong Prikry Property of $\mathbb{R}_u$ – details are provided in Gitik's handbook Chapter, Theorem 5.19.

(1) For each $1 \leq \tau < \kappa$, define

$$X_\tau = \{v \in \mathcal{U}_\infty \mid \ell(v) = \tau < \kappa_v\}$$

Note that $X_\tau \in u(\tau)$.

Therefore, $\bigcup_{1 \leq \tau < \kappa} X_\tau \in \mathcal{F}(u)$.

Now we "reflect" $\bigcup_{1 \leq \tau < \kappa} X_\tau$ considering

$$Y = \{v \in \mathcal{U}_\infty \mid \ell(v) < \kappa_v \wedge \bigcup_{1 \leq \tau < \ell(v)} (X_\tau \cap V_{\kappa_v}) \in \mathcal{F}(v)\}$$

Once again $Y \in \mathcal{F}(u)$.

By density, there is $p \in G$ s.t

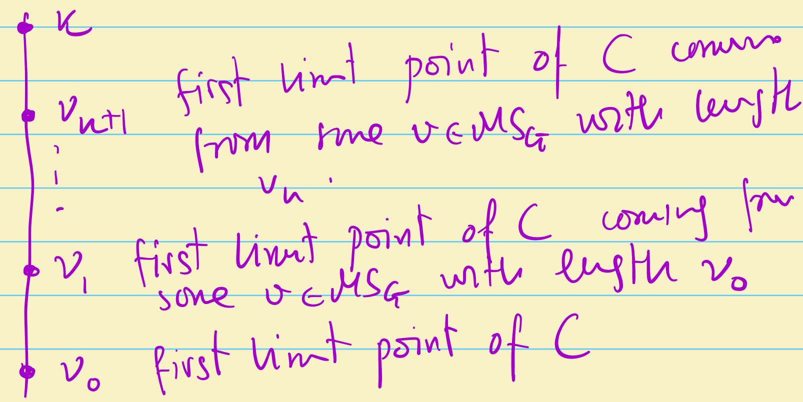
$$p = s ^\frown \langle (u, A) \rangle$$

for $A \subseteq Y$. Therefore, $C := C(G)$ is s.t

$$[\, C \setminus \max\{\kappa_v \mid v \in s\} \subseteq A \subseteq Y \,]$$

Let's now define an ω-seq converging to κ.

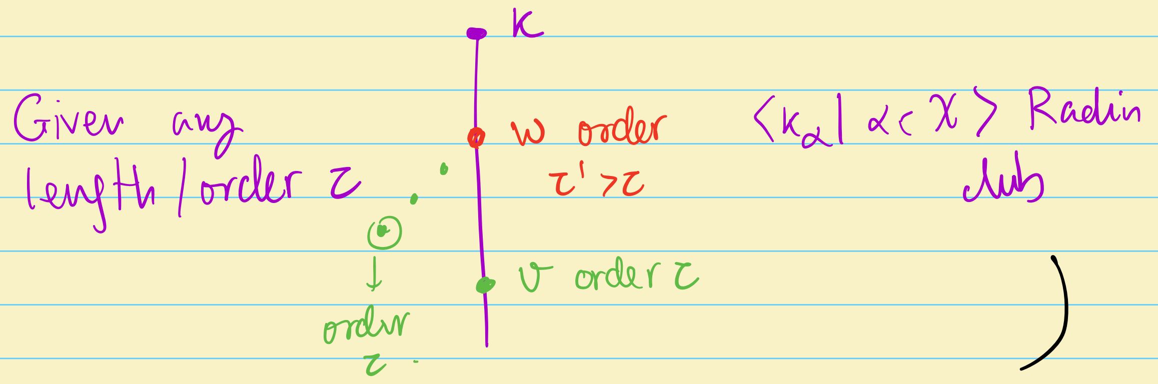
- $\nu_0 := \min(\mathrm{acc}(C))$
- $\nu_{n+1} = \min\{\nu \in \mathrm{acc}(C) \mid \nu > \nu_n \wedge \exists v \in MS_G \,(\kappa_v = \nu$ and $v \in X_{\nu_n}\}$



(The above definition is well-posed for two reasons:

(1) Given $\nu \in C$ there is a <u>unique</u> $v \in MS_G$ s.t. $\kappa_v = \nu$.

(2) $\{\kappa_v \mid v \in MS_G \wedge v \in X_\tau\}$ is unbounded in κ for all $\tau < \kappa$



)

We claim that $\nu_\omega := \sup_n \nu_n = \kappa$.

For suppose not. Since C is a club, $\nu_\omega \in C$.

Therefore there is $q \leq p$ in G mentioning the unique $v \in MS_G$ s.t $\kappa_v = \nu_\omega$.

Since $q \leq p$ it must be that $v \in A$ and so

$$\ell(v) < \nu_\omega \text{ and } \bigcup_{\tau < \ell(v)} (X_\tau \cap V_{\nu_\omega}) \in \mathcal{F}(v).$$

Therefore we may assume that

$$q = s_0 {}^\frown \langle (v, B) \rangle {}^\frown s_1$$

where $B \subseteq \bigcup_{\tau < \ell(v)} (X_\tau \cap V_{\nu_\omega})$.

Since $\ell(v) < \nu_\omega$ there is $n < \omega$ s.t $\ell(v) < \nu_n$.

and ν_n has not beeing mentioned in q.

Let $q' \in G$ s.t ν_n is mentioned in q'.

Then there is a unique $v_n \in MS_G$, $v_n \in X_\tau$

$\tau < \ell(\upsilon) < \nu_n$. Since this holds for all $\langle \nu_m \mid m \geq n \rangle$ it also holds for ν_{n+1}. However, ν_{n+1} was chosen from X_{ν_n} # $\square$

Mitchell's genericity criterion:

Theorem (Mitchell)

G is geometric for $\mathbb{R}_u \iff G$ is generic for $\mathbb{R}_u$. $\square$

Definition: G is geometric for $\mathbb{R}_u$ if:

(1) $G : \mathrm{dom}(G) \in \mathrm{Ord} \longrightarrow \mathcal{U}_\infty$ s.t

$$\{ \kappa_{G(\alpha)} \mid \alpha \in \mathrm{dom}(G) \}$$

is a club on κ_u.

(2) For each $\alpha \in \mathrm{dom}(G)$, $G \restriction \alpha$ is generic for $\mathbb{R}_{G(\alpha)}$

(3) $\mathcal{F}(u) = \{ X \subseteq V_{\kappa(u)} \mid \exists \alpha \, \forall \beta > \alpha \;\; G(\beta) \in X \}$. $\square$