Two days of Radin forcing

0. Introduction and historical context

Cohen: Invented forcing as a means to manipulate the power-set function $\kappa \mapsto 2^\kappa$ in models of ZFC.

Q: Can we use forcing to change other natural characteristics of cardinals? For example, the cofinality of a cardinal.

First (unsatisfactory) answer: Yes, simply collapse your favorite regular κ to some regular $\lambda < \kappa$.

Problem: Can you change cofinalities without collapsing cardinals?

Theorem (Jensen):

If $0^\#$ does not exist then every singular cardinal is singular in L.

Conclusion: Very large cardinals are needed to change cofinalities without collapsing cardinals.

In his PhD thesis showed that very large cardinals suffice to accomplish this:

Theorem (Prikry)

Suppose that κ is a measurable cardinal. Then there is a cardinal preserving forcing extension where κ remains a singular, strong limit cardinal with $\mathrm{cof}(\kappa)=\omega$. $\square$

Motivation: The Singular Cardinals Problem

Some relevant works in chronological order:

1) Building on earlier breakthroughs of Gödel and Cohen, Easton proved that the only ZFC restriction on

$$\kappa \in \mathrm{Reg} \mapsto 2^{\kappa}$$

are: (1) Monotonicity ; (2) König's Lemma.

2) Silver and Prikry combined supercompact cardinals with Prikry forcing to produce the first model of $\neg SCH_\kappa$ at a "high up" singular.

(3) In a two paper series, Magidor proved the consistency of

$$\forall n<\omega\ (2^{\aleph_n} = \aleph_{n+1}) + 2^{\aleph_\omega} = \aleph_{\omega+2}.$$

(4) Work of Mitchell, Woodin and Gitik finally pinned down the exact large cardinal hypothesis to get

$$\neg SCH_{\aleph_\omega}$$

1 A review of Prikry forcing:

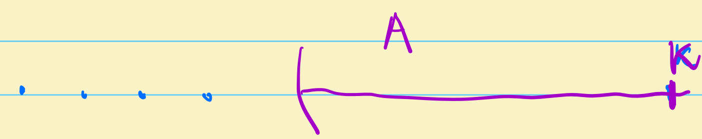
Fix a measurable cardinal κ and $\mathcal{U}$ a κ-complete normal ultrafilter on κ.

Def (Prikry forcing):

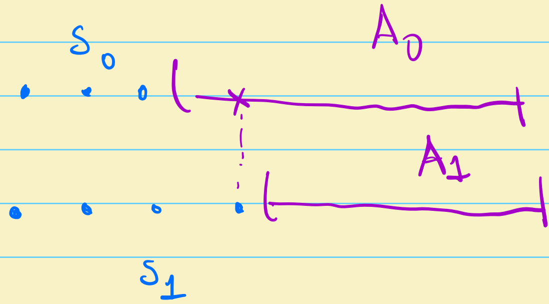
A condition in $\mathbb{P} := \mathbb{P}(\mathcal{U})$ is a pair (s, A) such that:

1) s is a finite increasing seq of ordinals $< \kappa$
(The stem)

2) $A \in \mathcal{U}$ and $\max(s) < \min(A)$.



Given conditions (s_1, A_1), (s_0, A_0) we write $(s_1, A_1) \leq (s_0, A_0)$ if $s_0 \sqsubseteq s_1$, $s_1 \setminus s_0 \subseteq A_0$ and $A_1 \subseteq A_0$.



The "Prikry" or "Pure extension" order of $\mathbb{P}$ (denoted $\leq^*$) is defined as:

$$(s_1, A_1) \leq^* (s_0, A_0) :\Longleftrightarrow s_0 = s_1 \wedge A_1 \subseteq A_0$$

$\boxtimes$

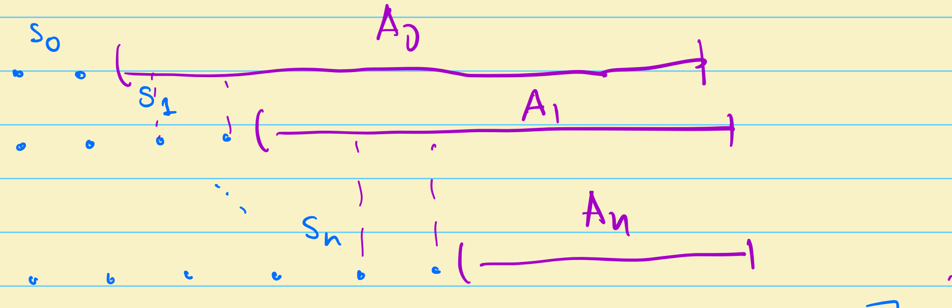
Note:

1) The order $\leq^*$ has much better closure properties than the main order $\leq$.

$\leq^*$ is κ-closed because $\mathcal{U}$ is κ-complete:

$$\langle (s, A_\alpha) \mid \alpha < \beta < \kappa \rangle \overset{*}{\geq} \Big(s, \bigcap_{\alpha<\beta} A_\alpha\Big)$$

2) Forcing with $\mathbb{P}$ introduces a cofinal set $C \subseteq \kappa$ with $\operatorname{otp}(C) = \omega$.



$$\Big[C = \bigcup \{ s^p \mid p \in G \} \Big]$$

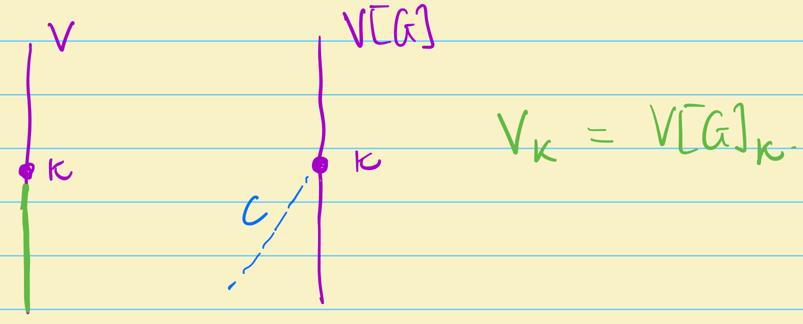
Theorem (Prikry):

The following hold for $\mathbb{P}$:

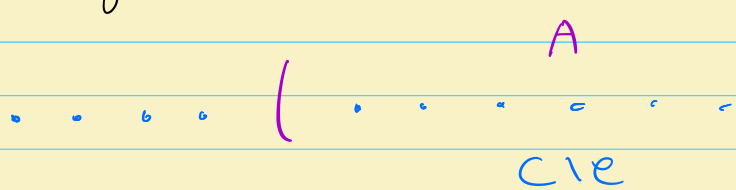
(1) $\mathbb{P}$ is κ^+-cc.

(2) $\mathbb{P}$ has the Prikry property; namely, given any $p \in \mathbb{P}$ and any sentence φ in the language of forcing of $\mathbb{P}$ there is $q \leq^* p$ s.t either $q \Vdash \varphi$ or $q \Vdash \neg\varphi$.

(3) $\mathbb{P}$ preserves cardinals and it does not add bounded subsets of κ.



(4) If $C \subseteq \kappa$ is the Prikry set then $C \subseteq^* A$ for all $A \in \mathcal{U}$



(5) (Mathias) The above characterizes Prikry seq.

2 A short overview of Magidor forcing

Magidor's forcing is a generalization of Prikry forcing that allows to change the cofinality of κ to any prescribed $\omega_1 \leq cof(\delta) = \delta < \kappa$.

Basic idea: A Magidor sequence consists of Magidor points with smaller order.
(A magidor point of order 1 is like the limit of a Prikry sequence.)

Example (Adding an ω^2-sequence):

Suppose that κ is a measurable cardinal and $\mathcal{U}_0 \triangleleft \mathcal{U}_1$ are κ-complete, normal uf on κ.
$(\mathcal{U}_0 \in Ult(V, \mathcal{U}_1))$

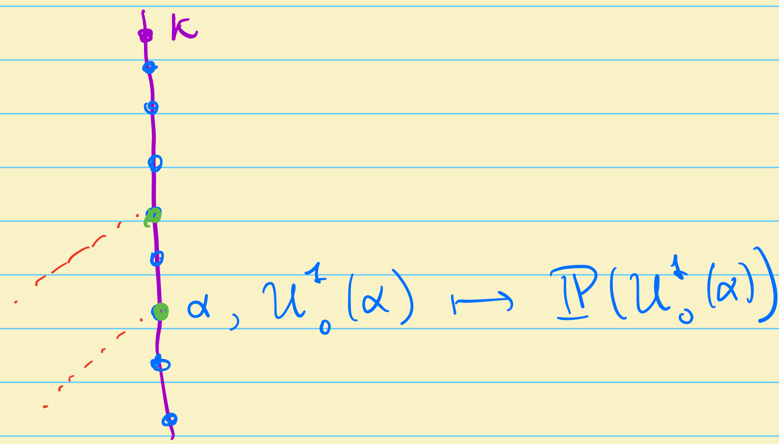
By definition, there is $\mathcal{U}_0^1 : \kappa \longrightarrow V$ s.t

$$\mathcal{U}_0 = j_{\mathcal{U}_1}(\mathcal{U}_0^1)(\kappa)$$

where $j_{\mathcal{U}_1} : V \to Ult(V, \mathcal{U}_1)$.

By Łos theorem,

$$\{\alpha < \kappa \mid \mathcal{U}^1_0(\alpha) \text{ is an } \alpha\text{-comp, normal uf on } \alpha\} \in \mathcal{U}_1$$



The idea behind Magidor's forcing $\mathbb{M}(\mathcal{U}_0, \mathcal{U}_1)$ is:

(1) We have points of two types: points of order 0 and points of order 1.

(2) Magidor's forcing generically chooses ω-many points of order 1 and then it forces with the corresponding Prikry procedure.

Example (Introducing an ω^3-sequence):

We start with $\mathcal{U}_0 \triangleleft \mathcal{U}_1 \triangleleft \mathcal{U}_2$

(1) We have points of order 0, points of order 1 and points of order 2.

(2)

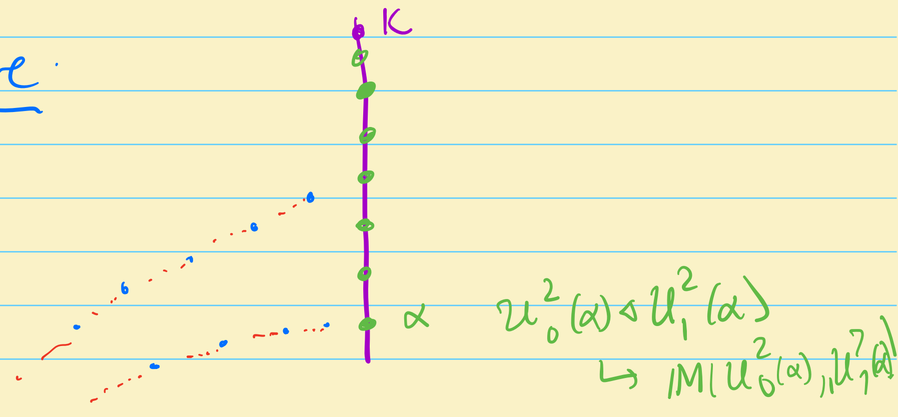
$$\mathcal{U}_0 = j_{\mathcal{U}_1}(\mathcal{U}_0^1)(\kappa)$$ $\equiv$ representation of $\mathcal{U}_0$ in $M_{\mathcal{U}_1}$

$$\mathcal{U}_0 = j_{\mathcal{U}_2}(\mathcal{U}_0^2)(\kappa)$$ $\equiv$ " " of $\mathcal{U}_0$ in $M_{\mathcal{U}_2}$

$$\mathcal{U}_1 = j_{\mathcal{U}_2}(\mathcal{U}_1^2)(\kappa)$$

This leads to the "fractal-like structure" of Magidor's forcing and to "coherency".

Fractal-like structure:



We know that

$$\{\alpha<\kappa \mid \mathcal{U}^2_0(\alpha) \triangleleft \mathcal{U}^2_1(\alpha)\} \in \mathcal{U}_2$$

<u>Coherency</u>:

$$\mathcal{U}_2 \ni \{\alpha<\kappa \mid \{\beta<\kappa \mid \mathcal{U}^2_0(\alpha) = j_{\mathcal{U}^2_1(\alpha)}(\mathcal{U}^1_0)(\beta)\} \in \mathcal{U}_1\}$$

<u>Theorem (Magidor)</u>:

Suppose that κ is a measurable carrying a $\triangleleft$-inc $\langle \mathcal{U}_\alpha \mid \alpha<\delta\rangle$ with $\omega_1 \leq \mathrm{cof}(\delta) = \delta < \kappa$.

Then there is a cardinal-preserving extension where

(1) κ is strong limit

(2) There is $C \subseteq \kappa$ closed and unbounded with $\mathrm{otp}(C) = \omega^\delta$.

(In particular, $\Vdash_{\mathbb{M}(\vec{\mathcal{U}})}$ "$\mathrm{cof}(\kappa) = \mathrm{cof}(\delta)$".)

3 Radin forcing:

Key Goal: Being able to:

(1) Generalize Magidor's forcing.

(2) Add a club $C \subseteq \kappa$ consisting of former regulars (measurables, etc) while keeping κ regular (or measurable, or strongly compact ....).

$\boxtimes$

Why useful?: Here is an easy application:

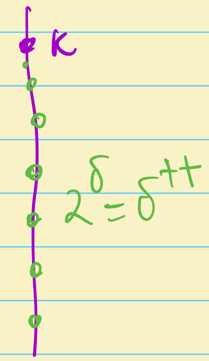
- Start with κ a supercompact cardinal with $2^{\kappa} = \kappa^{++}$.

(By "reflection" there is a "measure one" set A where $2^{\delta} = \delta^{++}$, for all $\alpha \in A$)

- Shoot a Radin club $C \subseteq \kappa$ through A (i.e $C \subseteq A$) while making sure that: κ remains regular.

- In this way we get a model $V[G]$ where

$2^\delta = \delta^{++}$ for all $\delta \in C$ and κ is innaccessible



- Cutting $V[G]$ at κ we get:

  (a) $V[G]_\kappa \models ZFC$

  (b) $\langle V[G]_\kappa, \in, C \rangle$

  $\models$ "C is closed unbounded in Ord $\wedge \forall \delta \in C \ 2^\delta \neq \delta^+$".

3.1 The basics of Radin forcing:

The basic building block of Radin forcing are "constructing embeddings" which play the analogous role of $\langle U_\alpha \mid \alpha < \delta \rangle$ in Magidor's.

Def: Suppose that $j: V \to M$ is an elementary embedding with $cp(j) = \kappa$. Define u^j as follows:

$$u^j = \langle u^j(\alpha) \mid \alpha < \ell(u^j) \rangle$$

where:

- $u^j(0) := \kappa$

- $u^j(1) := \{ X \subseteq V_\kappa \mid \langle \kappa \rangle \in j(X) \}$

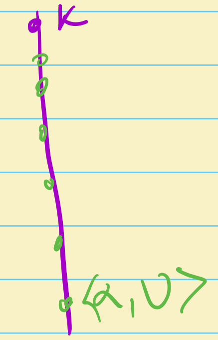
(This is RK-equivalent to the normal measure on κ.)

$\left(u^j(1) \ni \{ \langle \alpha \rangle \mid \alpha \text{ is inaccc} \wedge \alpha < \kappa \} \right)$

- $u^j(\alpha) := \{ X \subseteq V_\kappa \mid u^j \restriction \alpha \in j(X) \}$.

$\left(u^j(2) = \{ X \subseteq V_\kappa \mid \underbrace{\langle \kappa, u^j(1) \rangle}_{u^j \restriction 2} \in j(X) \} \right.$

$u^j(2) \ni \underbrace{\{ \langle \alpha, U \rangle \mid \alpha < \kappa \wedge U \text{ is an } \alpha\text{-complete uf on } V_\alpha \}}_{X_{1,\kappa}}$



$u^j(3) = \{ X \subseteq V_\kappa \mid \langle \kappa, u^j(1), u^j(2) \rangle \in j(X) \}$

$u^j(3) \ni \{ \langle \alpha, U, V \rangle \mid \langle \alpha, U \rangle \in X_1$ and

V is an α-complete ultrafilter

$X_{1,\alpha} = \{ \langle \beta, W \rangle \mid \beta < \alpha \wedge W \text{ is a } \beta\text{-complete ultr on } V_\beta \}$.