Two closed graphs with uncountable different Borel chromatic numbers

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Definition

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- The Γ-chromatic number of G, denoted by $\chi_\Gamma(G)$, is the least κ for which there exists a κ-coloring of G which is Γ-measurable.

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- See [Mil08] for more.

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The G_0-graph is the graph on 2^ω defined by

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For proof see (see [KST99].

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The G_0-dichotomy implies many known dichotomies in descritive set theory such as the perfect set property or the Silver's dichotomy on co-analytic equivalence relations (see [Mil12]).

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This is a situation where the Borel chromatic number of all analytic graphs is either countable or as big as the continuum. We will first see that this is independent of ZFC.

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In this model, the continuum is $2^{\aleph_0} = \aleph_2$. It remains open if the same holds for the classe of analytic graphs on Polish spaces.

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We answer this question affirmatively.

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- $\chi_\mu(G_1) \geq \mathrm{cov}(\mathcal{N})$
- $\chi_\mu(G_0) = 3$ (see [Mil08]).

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- We proved that every random real is contained in a Borel G_0-independet set coded in the ground model.
- We do not know whether the same can be said about any other real added in the random extension.

Definition

- A tree $p \subseteq 2^{<\omega}$ is perfect iff for every $t \in p$ there is $s \in p$ such that $t \subseteq s$ and s is a splitting node of p — i.e., $s^\frown 0, s^\frown 1 \in p$.

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Remark It is the same as the forcing notion of parcial function from ω to 2 with co-infinite domain ordered by direct inclusion.

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- Let $p \subseteq 2^{<\omega}$ be a perfect tree. For $n \in \omega$ we denote by $\mathrm{split}_n(p)$ the set of all $t \in p$ that are minimal in p with respect to $\subseteq$ such that below t there are exactly n proper splitting nodes in p.

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- We say that a sequence $(p_n)_{n\in\omega}$ of Sacks (Silver) conditions is a fusion sequence for the Sacks (respec. Silver)forcing iff

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It should be noted that if If $(p_n)_{n\in\omega}$ is a fusion sequence for the Sacks or Silver forcing then $q \doteq \bigcap_{n\in\omega} p_n$ is a Sacks or a Silver condition.

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$$\aleph_1 = \chi_B(G_0) < \chi_B(G_1) = \aleph_2 = 2^{\aleph_0}.$$

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For $\mathbb{P}$ forcing notion and $\dot{x}$ a $\mathbb{P}$-name for an element of ω^ω wtinessed by p, define for each $q \leq p$,

$$T_q(\dot{x}) = \{s \in \omega^{<\omega} \mid \exists r \leq q(r \Vdash s \subseteq \dot{x})\},$$

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$$q * \sigma = \{s \in q \mid s \subseteq q_\sigma \vee q_\sigma \subseteq s\} \in \mathbb{V}.$$

For a splitting level n, define the Silver tree

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This helps us to handle the successor case for the property of not adding G_0-independent closed sets. We need to import the fusion technology to the iterations:

Let F be a finite subset of α and $\eta : F \to \omega$. For $p, q \in \mathbb{V}_\alpha$ let

$$p \leq_{F,\eta} q \leftrightarrow \forall \gamma \in F(p \restriction \gamma \Vdash p(\gamma) \leq_{\eta(\gamma)} q(\gamma)).$$

A sequence $(p_n)_{n\in\omega}$ is a fusion sequence if there are an increasing sequence $(F_n)_{n\in\omega}$ of finite subsets of α and a sequence $(\eta_n : F_n \to \omega \mid n \in \omega)$ satisfying that for all $n \in \omega$

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The fusion p_ω of a fusion sequence $(p_n)_{n\in\omega}$ in $\mathbb{V}_\alpha$ is defined recursively by

$$p_\omega \restriction \gamma \Vdash p_\omega(\gamma) = \bigcap_{n\in\omega} p_n(\gamma).$$

Let $p \in \mathbb{V}_\alpha$, F a finite subset of α and $\eta : F \to \omega$. For $\sigma \in \prod_{\gamma \in F} 2^{\eta(\gamma)}$ we define a condition $p * \sigma \in \mathbb{V}_\alpha$ by

$$(p * \sigma) \restriction \delta \Vdash (p * \sigma)(\delta) = p(\delta) * \sigma(\delta)$$

if $\delta \in F$, and $(p * \sigma)(\delta) = p(\delta)$ if $\delta \in \alpha \setminus F$.

For F a finite subset of α and $\eta : F \to \omega$, the right notion of faithfulness for a condition $p \in \mathbb{V}_\alpha$ can give provide us with various preservation theorems in the Silver model.

For F, η as above

(a) we can say that $p \in \mathbb{V}_\alpha$ is (F, η)-faithfull iff for every $\sigma, \tau \in \prod_{\gamma \in F} 2^{\eta(\gamma)}$ such that $\sigma \neq \tau$, $\dot{x}_{p*\sigma}$ and $\dot{x}_{p*\tau}$ — the maximal initial segments of $\dot{x}$ decided by $p * \sigma$ and $p * \tau$, respectively — are incompatible.

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In the first case we get the preservation of the 2-localization property. In the second case we get the preservation of not adding G_0-independent closed sets.

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For the part (a), it is useful to note that for any Silver tree p, there is $q \leq p$ such that $[q]$ is G_0-independent.

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For the part (a), it is useful to note that for any Silver tree p, there is $q \leq p$ such that $[q]$ is G_0-independent. This means that the Silver real is always in some closed G_0-independent ground model set.

Theorem

Let α be an ordinal, $\mathbb{V}_{\alpha+1}$ an $\alpha+1$ countable supported iteration of copies of the Silver forcing and and let $\dot{x}$ be a $\mathbb{V}_{\alpha+1}$-name for an element of 2^ω only added at stage $\alpha+1$ and $p \in \mathbb{V}_{\alpha+1}$ such that $p \Vdash \dot{x} : \omega \to 2$ witnessing that.

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Using the previows lemma and the maximal principle, there are $\mathbb{V}_\alpha$-names $\dot{q}$ for a condition and $\dot{h}$ a homeomorphism such that $p \restriction \alpha$ forces

- $\dot{q} \leq p(\alpha)$,
- $\dot{h} : \dot{q} \to [T_{\dot{q}}(\dot{x})]$ maps the generic real onto $\dot{x}$, and either
  - (a) $T_{\dot{q}}(\dot{x})$ is a Silver tree such that $[T_{\dot{q}}(\dot{x})]$ is G_0-independent, or
  - (b) any two incompatible nodes of $T_{\dot{q}}(\dot{x})$ disagree in at least two coordinates.

Theorem

Let α be a limit ordinal, $\mathbb{V}_\alpha$ an α countable supported iteration of copies of the Silver forcing and let $\dot{x}$ be a $\mathbb{V}_\alpha$-name for an element of 2^ω only added at stage α and $p \in \mathbb{V}_\alpha$ such that $p \Vdash \dot{x} : \omega \to 2$ witnessing that.

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In particular, no Silver reals are added at limit steps of countable supported iterations of the Silver forcing.

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