In this note we prove the following result:

Proposition 1. *Let $\langle L, <\rangle$ be a linear order, α an ordinal, and suppose there is an injection $i : L \hookrightarrow {}^{\alpha}2$. Then $\langle L, <\rangle \not\to (\eta)^{\eta}$.*

Equivalently, if $\langle L, <\rangle$ is a linear order with $\langle L, <\rangle \to (\eta)^{\eta}$, then the set L witnesses the failure of the Kinna-Wagner principle KWP_1.

We fix throughout a linear order $\langle L, <\rangle$, an ordinal α, and an injection $i : \langle L, <\rangle \hookrightarrow {}^{\alpha}2$. Our proof of Proposition 1 involves a complicated definition of a colouring of $[\langle L, <\rangle]^{\eta}$ with no homogeneous set, for which we will first need some definitions and notation.

Definitions and notation

We write η for the order type of the rationals under their usual ordering. An order is *scattered* if it has no suborder ordered as η; otherwise, it is *non-scattered*. A countable order is non-scattered iff is it bi-embeddable with the rationals.

For α an ordinal, ${}^{\alpha}2$ is topologised by the basic open sets $[s] \coloneqq \{x \in {}^{\alpha}2 : s \sqsubseteq x\}$ for $s \in {}^{<\alpha}2$.

For α a fixed ordinal and $X \subseteq {}^{\alpha}2$, we say that $s \in {}^{<\alpha}2$ is *η-splitting* for X if

$$X_s^- \coloneqq [s^\frown\langle 0\rangle] \cap X = \{x \in X : s^\frown\langle 0\rangle \sqsubseteq x\} \text{ and}$$
$$X_s^+ \coloneqq [s^\frown\langle 1\rangle] \cap X = \{x \in X : s^\frown\langle 1\rangle \sqsubseteq x\}$$

are both non-scattered as suborders of $\langle {}^{\alpha}2, <_{\text{lex}}\rangle$.

Let $\langle L, <\rangle$ a linear order, α an ordinal, and $i : L \hookrightarrow {}^{\alpha}2$ an injection be fixed as above, and let $A \subseteq L$. For $a \in A$, write

$$\rho_A(a) \coloneqq \min\{\beta \leq \alpha : \forall b \in A, a \neq b \implies i(a) \restriction \beta \neq i(b) \restriction \beta\}.$$

We note that $\rho_A(a)$ always exists as e.g. $\beta = \alpha$ has the property described. This $\rho_A(a)$ can be thought of as a measure of how isolated the point $i(a)$ is in $i\,``\,A \subseteq {}^{\alpha}2$, according to the topology described above; $i(a)$ is isolated in $i\,``\,A$ iff $\rho_A(a) < \alpha$.

By extension, for $A \subseteq L$ we write

$$\rho(A) \coloneqq \sup_{a \in A} \rho_A(a).$$

This $\rho(A)$ is the least ordinal $\beta \leq \alpha$ such that all members of $i\,``\,A$ can be distinguished by their restriction to β.

For $A \subseteq L$, $\delta \leq \alpha$, we write

$$A_\delta \coloneqq \{a \in A : \rho_A(a) = \delta\},$$
$$A_{\leq\delta} \coloneqq \{a \in A : \rho_A(a) \leq \delta\} = \bigcup_{\gamma \leq \delta} A_\gamma, \text{ and}$$
$$A_{>\delta} \coloneqq \{a \in A : \rho_A(a) > \delta\} = A \setminus A_{\leq\delta}.$$

We refer to each A_δ as the δ^{th} *level* of A.

Preliminary observations

1. For any X, α with $X \subseteq {}^{\alpha}2$, if $s_0, s_1 \in {}^{<\alpha}2$ are both η-splitting for X, then so is their maximal common initial segment $\delta(s_0, s_1)$, and so if $X \subseteq {}^{\alpha}2$ has any η-splitting nodes, it has a unique one of minimal length.

2. Let $A \in [\langle L, <\rangle]^\eta$. For any $B \subseteq i\,``\,A$, there is $A' \in [A]^\eta$ such that either $i\,``\,A' \subseteq B$ or $i\,``\,A' = (i\,``\,A) \setminus B$. This is because either $i^{-1}(B) \subseteq A$ is non-scattered as a suborder of $\langle L, <\rangle$, in which case any $A' \in [i^{-1}(B)]^\eta$ has $i\,``\,A' \subseteq B$, or $i^{-1}(B)$ is scattered, in which case $A' \coloneqq A \setminus (i^{-1}(B))$ is still ordered as $\eta$ and has $i\,``\,A' = (i\,``\,A) \setminus B$.

3. Let $B \subseteq A \subseteq L$ and let $a \in B$. Then $\rho_B(a) \leq \rho_A(a)$. In particular, it follows that $\rho(B) \leq \rho(A)$.

4. For any $A \subseteq L$, $\delta \leq \alpha$, B with $A_{>\delta} \subseteq B \subseteq A$ and $a \in A_{>\delta}$,
$$\rho_A(a) = \rho_B(a).$$
In other words, the levels of A strictly above δ are preserved if we remove elements of A whose level is at most δ. To see this, let $a \in A_{>\delta}$, so $\rho_A(a) = \gamma > \delta$ for some γ; then for any $\beta \in [\delta, \gamma)$ there is some $b_\beta \in A$, $b \neq a$, with $i(a) \restriction \beta = i(b) \restriction \beta$. But then this b necessarily has $\rho_A(b) \geq \beta$, so $b \in A_{>\delta}$, and in particular $b \in B$. It follows that $\rho_B(a) \geq \rho_A(a)$, and by the previous observation we conclude that $\rho_B(a) = \rho_A(a)$.

The colouring

Proof of Proposition 1. We shall define a colouring $c : [\langle L, <\rangle]^\eta \to 2$ with no homogeneous set by means of a number of cases. For $A \in [\langle L, <\rangle]^\eta$, we first consider the order type of $i\,``\,A \subseteq {}^{\alpha}2$, equipped with the induced suborder which it inherits from $\langle {}^{\alpha}2, <_{\text{lex}}\rangle$. This can be any countable order type. If we can sufficiently easily switch between these by reducing A to some $A' \in [\langle L, <\rangle]^\eta$, we can exploit this in how we define our colouring:

Case 1: There are A', $A'' \in [A]^\eta$ such that $i\,``\,A'$ is ordered as $\eta$ and $i\,``\,A''$ is not ordered as η.

In this case, we set $c(A) = 0$ if $i\,``\,A$ is ordered as $\eta$, and $c(A) = 1$ otherwise. If we can always reduce from an $A \in [\langle L, <\rangle]^\eta$ to some $A' \in [A]^\eta$ such that precisely one of $i\,``\,A$, $i\,``\,A'$ is ordered as $\eta$, then $c$ cannot have a homogeneous set; it remains, therefore, to deal with those $A \in [\langle L, <\rangle]^\eta$ which do not have this property, so assume for the rest of the proof that $A$ is either such that $i\,``\,A$ is not ordered as η and there is no $A' \in [A]^\eta$ with $i\,``\,A'$ ordered as $\eta$, or that $i\,``\,A$ is ordered as η, and so is $i\,``\,A'$ for every $A' \in [A]^\eta$. $\dashv_{\text{Case 1}}$

Case 2: $i\,``\,A$ is ordered as $\eta$, and so is $i\,``\,A'$ for every $A' \in [A]^\eta$.

Observe that for any interval $B \subsetneq i\,``\,A$ which is not empty and not a singleton, we have that $i^{-1}(B)$ is non-scattered; if $i^{-1}(B)$ were scattered, then we could remove all but two of its elements from $A$ to obtain $A' \in [A]^\eta$ with the property that some two elements of $i\,``\,A'$ have no element of $i\,``\,A'$ between them, contradicting our assumption that i " A' is ordered as η for all $A' \in [A]^\eta$. In particular, for any $s \in {}^{<\alpha}2$, $[s] \cap i$ " A has non-scattered preimage under i unless it is empty or a singleton. We can now apply the same colouring used to show that $\langle {}^\alpha 2, <_{\text{lex}} \rangle \not\to (\eta)^\eta$; we describe this in detail in the following paragraph.

For $A \in [\langle L, < \rangle]^\eta$ in Case 2, associate A with three nodes $s_A, s_A^0, s_A^1 \in {}^{<\alpha}2$ in the following way: s_A is the unique minimal-length η-splitting node for i " $A \subseteq {}^\alpha 2$, s_A^0 is the unique minimal-length η-splitting node for $[s^\frown\langle 0 \rangle] \cap (i$ " $A)$, and s_A^1 is the unique minimal-length η-splitting node for $[s^\frown\langle 1 \rangle] \cap (i$ " $A)$. Now we set

$$c(A) = \begin{cases} 0 & \text{if } \operatorname{len}(s_A^0) \geq \operatorname{len}(s_A^1); \\ 1 & \text{if } \operatorname{len}(s_A^0) < \operatorname{len}(s_A^1). \end{cases}$$

Similarly to the argument that $\langle {}^\alpha 2, <_{\text{lex}} \rangle \not\to (\eta)^\eta$, we now show that there can be no homogeneous set for this colouring by means of the following claim:

Claim 1. For A in Case 2, if $t_0, t_1 \in {}^{<\alpha}2$ are both η-splitting for i " A and $t_0 <_{\text{lex}} t_1$, then there is some $A' \in [A]^\eta$ with $s_{A'}^0 = t_0$ and $s_{A'}^1 = t_1$.

Proof of claim: Fix A, t_0, t_1. It suffices to find $A' \in [A]^\eta$ with i " $A' \subseteq [t_0] \cup [t_1]$ and i " $A' \cap [t_j^\frown\langle k \rangle]$ non-scattered for $j, k \in \{0, 1\}$ (for which it in fact suffices to ensure that i " $A' \cap [t_j^\frown\langle k \rangle]$ has at least two elements). By definition, each i " $A \cap [t_j^\frown\langle k \rangle]$ is non-scattered, so has non-scattered preimage under i by the observation above; it is easy to check that we can find subsets of $A \cap \bigcup_{j,k \in \{0,1\}} i^{-1}([t_j^\frown\langle k \rangle])$ ordered as η whose intersection with each $i^{-1}([t_j^\frown\langle k \rangle])$ is also ordered as η. Take A' to be any such subset. $\blacksquare_{\text{Claim 1}}$

It now follows exactly as in the proof that $\langle {}^\alpha 2, <_{\text{lex}} \rangle \not\to (\eta)^\eta$ that no A in Case 2 can be homogeneous for c. $\dashv_{\text{Case 2}}$

Cases 1 and 2 together deal with those $A \in [\langle L, < \rangle]^\eta$ for which there is some $A' \in [A]^\eta$ with i " A' ordered as η, so for the rest of the proof we may assume that this is false for every A which we consider.

We may further assume that $\rho(A)$ is minimal in $\{\rho(A') : A' \in [A]^\eta\}$. This is because if $A \in [\langle L, < \rangle]^\eta$ is homogeneous for c, then any $A' \in [\langle L, < \rangle]^\eta$ is also homogeneous, and so we can reduce to some A' with $\rho(A')$ minimal in this sense. The value of $c(A)$ for those A with $\rho(A) \neq \min\{\rho(A') : A' \in [A]^\eta\}$ is therefore irrelevant; we may define it arbitrarily, e.g. $c(A) = 0$ for all such A. This assumption on A has the following two important consequences:

Claim 2. Let $A \in [\langle L, < \rangle]^\eta$ be such that $\rho(A) = \min\{\rho(A') : A' \in [A]^\eta\}$. Then:

(a) If $\delta < \rho(A)$, then $A_{\leq \delta}$ is scattered as a suborder of $\langle L, < \rangle$;

(b) $\rho(A)$ is a limit ordinal.

Proof of claim:

(a) If $\delta < \rho(A)$ is such that $A_{\leq \delta}$ is non-scattered, then there is some $A' \in [A_{\leq \delta}]^\eta$, but then $\rho(A') \leq \delta < \rho(A)$, contradicting the minimality of $\rho(A)$.

(b) Suppose $\rho(A) = \beta' + 1$ for some β'. Then we have that at least one of

$$A^0 := \{a \in A : i(a)(\beta') = 0\};$$
$$A^1 := \{a \in A : i(a)(\beta') = 1\}$$

is non-scattered, from which it follows that there is some $A' \in [A]^\eta$ such that either $A' \subseteq A^0$ or $A' \subseteq A^1$. In either case, $\rho(A') \leq \beta' < \rho(A)$, contradicting the minimality of $\rho(A)$. $\blacksquare_{\text{Claim 2}}$

Let us write $\beta := \rho(A) = \min\{\rho(A') : A' \in [A]^\eta\}$. We split into cases according to whether $A_{<\beta}$ is empty or not.

Case 3: There are $A', A'' \in [A]^\eta$ with $A'_{<\beta} \neq \emptyset$ and $A''_{<\beta} = \emptyset$.

In this case we simply set

$$c(A) = \begin{cases} 0 & \text{if } A_{<\beta} = \emptyset; \\ 1 & \text{if } A_{<\beta} \neq \emptyset. \end{cases}$$

$\dashv_{\text{Case 3}}$

Case 4: For all $A' \in [A]^\eta$, $A'_{<\beta} = \emptyset$.

In this case, $i\,"\,A$ is non-scattered as a suborder of $\langle {}^\alpha 2, <_{\text{lex}}\rangle$, as in particular every condensation class of $i\,"\,A$ has at most two elements. Here we use the same idea as in Case 2; write s_A to be the minimal-length η-splitting node for $i\,"\,A$, s^0_A the unique minimal-length η-splitting node for $[s^\frown\langle 0\rangle] \cap (i\,"\,A)$, and s^1_A the unique minimal-length η-splitting node for $[s^\frown\langle 1\rangle] \cap (i\,"\,A)$. Now we set

$$c(A) = \begin{cases} 0 & \text{if } \operatorname{len}(s^0_A) \geq \operatorname{len}(s^1_A); \\ 1 & \text{if } \operatorname{len}(s^0_A) < \operatorname{len}(s^1_A). \end{cases}$$

Claim 3. For A in Case 5, if $t_0, t_1 \in {}^{<\alpha}2$ are both η-splitting for $i\,"\,A$ and $t_0 <_{\text{lex}} t_1$, then there is some $A' \in [A]^\eta$ with $s^0_{A'} = t_0$ and $s^1_{A'} = t_1$.

Proof of claim: We will conclude this proof in the same way as the proof of Claim 1, but we need a different argument to show that the $[t_j^\frown\langle k\rangle] \cap i\,"\,A$ have non-scattered preimages. Observe by definition of $\rho(A)$ that for $t \in {}^{<\alpha}2$ to be η-splitting for $i\,"\,A$, we in fact must have $t \in {}^{<\beta}2$. Now suppose some $t \in {}^{<\alpha}2$ is such that $[t] \cap i\,"\,A$ is non-empty but scattered and let $a \in i^{-1}([t] \cap i\,"\,A)$. Then $A' := (A \setminus i^{-1}([t] \cap i\,"\,A)) \cup \{a\}$ is ordered as η, as we have only removed a scattered set from A, but $\rho_{A'}(a) \leq \operatorname{len}(t) < \beta$, contradicting our assumption that $A'_{<\beta}$ is empty for every $A' \in [A]^\eta$. Now we can proceed exactly as in the proof of Claim 1. $\blacksquare_{\text{Claim 3}}$

It follows as in the proof that $\langle {}^\alpha 2, <_{\text{lex}}\rangle \not\to (\tau)^\tau$ for any countable non-scattered τ that any A in Case 5 cannot be homogeneous for c. $\dashv_{\text{Case 4}}$

Case 5: For every $A' \in [A]^\eta$, $A'_{<\beta}$ is non-empty.

For any A in Case 5, we have in particular that $A_{<\beta}$ is infinite. It is helpful in this case to consider the projection of $i\,"\,A$ to ${}^{<\beta}2$ (i.e. by replacing each $i(a)$ by $i(a) \restriction \beta$; this is injective, as $\beta = \rho(A)$), so $A_{<\beta}$ and A_β correspond to isolated points and limit points, respectively. We now consider the sequence of i-preimages of Cantor-Bendixson derivatives of $i\,''\,A$ in ${}^{<\beta}2$:

$$\begin{aligned} A^{(0)} &:= A, \\ A^{(\xi+1)} &:= A^{(\xi)}_{\beta} \text{ for any ordinal } \xi, \text{ and} \\ A^{(\gamma)} &:= \bigcap_{\xi<\gamma} A^{(\xi)} \text{ for } \gamma \text{ a limit ordinal.} \end{aligned}$$

Let $\varphi_A < \omega_1$ be minimal such that $A^{(\varphi_A)}$ is not ordered as η. Reducing if necessary, assume that $\varphi_A = \min\{\varphi_{A'} : A' \in [A]^\eta\}$. Now, $A \setminus A^{(\varphi_A)}$ is necessarily non-scattered; any $A' \in \left[A \setminus A^{(\varphi_A)}\right]^\eta$ has the property that $(A')^{(\varphi_A)} = \emptyset$. Such an A' also necessarily has $\varphi_{A'} = \varphi_A$, by minimality of φ_A, so the sequence $\langle (A')^{(\xi)} : \xi < \omega_1 \rangle$ has the property that every term is ordered as η until $(A')^{(\varphi_A)}$, which is empty.

In this way we can, by reducing A to some $A' \in [A]^\eta$ if necessary, assume that the first term of the sequence $A^{(\xi)}$ not ordered as η is empty, that this is also true of all $A' \in [A]^\eta$, and that for all $A' \in [A]^\eta$ it happens at the same point in the sequence (i.e. at $\xi = \varphi_A$). We will assume that this is the case for all A which we consider for the rest of the proof, and define $c(A)$ arbitrarily for A in Case 5 not satisfying these further assumptions.

We note here two important consequences of these assumptions. Since φ_A is minimal, we have that for any $\xi < \varphi_A$, $A \setminus A^{(\xi)}$ is scattered. Otherwise, we could reduce to some $A' \in \left[A \setminus A^{(\xi)}\right]^\eta$, which would have $\varphi_{A'} \leq \xi < \varphi_A$, contradicting the minimality of φ_A. Since $A^{(\varphi_A)} = \emptyset$, for every $a \in A$ there is a unique ordinal $\psi_A(a) < \varphi_A$ with $a \in A^{(\psi_A(a))} \setminus A^{(\psi_A(a)+1)}$.

Now we will show that in this setting, we can essentially "pick out" any two elements of A; our colouring will ask whether the ordering of these two elements in $\langle L, < \rangle$ agrees with the ordering of their images in $\langle {}^\alpha 2, <_{\text{lex}} \rangle$, and have no homogeneous set because there will always be pairs for which the orderings agree and pairs for which they disagree.

Claim 4. Let A be in Case 5 and have the additional properties that $A^{(\varphi_A)} = \emptyset$ and that φ_A is minimal among $\{\varphi_{A'} : A' \in [A]^\eta\}$. Then for any $a, b \in A$, there is an $A' \in [A]^\eta$ and some $\delta < \beta$ such that $A'_{\leq\delta} = \{a, b\}$.

Proof of claim: Fix $a, b \in A$. We first reduce to some $A'' \in [A]^\eta$ such that $a, b \in A''_{<\beta}$. To do this, let $\psi := \max\{\psi_A(a), \psi_A(b)\}$ and set $A'' := A^{(\psi)} \cup \{a, b\}$. Then $\rho_{A''}(a) < \beta$ and $\rho_{A''}(b) < \beta$, for the following reasons: a and b are already "isolated points" in $A^{(\psi_A(a))}$ and $A^{(\psi_A(b))}$, respectively; $A'' \setminus \{b\} \subseteq A^{(\psi_A(a))}$ and $A'' \setminus \{a\} \subseteq A^{(\psi_A(b))}$; β is a limit ordinal so the common initial segment of a and b has length strictly less than β.

Now let $\delta := \max\{\rho_{A''}(a), \rho_{A''}(b)\}$, and set $A' := \{a, b\} \cup A''_{>\delta}$. By the minimality of $\rho(A)$, we have $\rho(A'') = \rho(A)$, and in particular, by the minimality of $\rho(A'')$ among $\{\rho(A') : A' \in [A'']^\eta\}$, $A''_{\leq\delta}$ is scattered. Since $A'' \setminus A' \subseteq A''_{\leq\delta}$, it follows that A' is ordered as η. Now, by preliminary observation 4, $A'_{>\delta} = A''_{>\delta}$, and since $\rho_{A'}(a) \leq \rho_{A''}(a) \leq \delta$ and $\rho_{A'}(b) \leq \rho_{A''}(b) \leq \delta$, it follows that $A'_{\leq\delta} = \{a, b\}$. $\blacksquare_{\text{Claim 4}}$

In this way we can essentially "pick out" any two elements of A. We now define a colouring based on whether the ordering of these two elements in $\langle L, <\rangle$ agrees with the ordering of their images in $\langle {}^{\alpha}2, <_{\text{lex}}\rangle$ or not. For A in Case 5, define $c(A)$ arbitrarily on those A for which there is no $\delta < \beta$ with $|A_{\leq\delta}| = 2$, and for those A such that $|A_{\leq\delta}| = 2$ for some δ,

$$c(A) = \begin{cases} 0 & \text{if } i \text{ is order-preserving on } A_{\leq\delta}; \\ 1 & \text{if } i \text{ is order-reversing on } A_{\leq\delta}. \end{cases}$$

Then since by assumption $i\,``\,A$ is not ordered as η, it is necessarily the case both that there are $a < b$ in A with $i(a) < i(b)$ and that there are $a' < b'$ in A with $i(a') > i(b')$; applying Claim 2, it follows that there are $A', A'' \in [A]^\eta$ with $c(A') \neq c(A'')$. $\dashv_{\text{Case 5}}$

□