Proposition 1. *Let τ be an order type with $\tau+\tau \le \tau$. Then for any ordinal α, $\langle {}^{\alpha}2, <_{lex}\rangle \not\to (\tau)^{\tau}$.*

Proof. For $A \subseteq {}^{\alpha}2$, $s \in {}^{<\alpha}2$, write

$$A^0_s := \{x \in A : s^\frown\langle 0\rangle \sqsubseteq x\}$$
$$A^1_s := \{x \in A : s^\frown\langle 1\rangle \sqsubseteq x\}.$$

For $A \in [\langle {}^{\alpha}2, <_{\text{lex}}\rangle]^{\tau}$, $s \in {}^{<\alpha}2$, say s is *τ-splitting for A* if A^0_s and A^1_s both embed τ.

Claim 1. For any $A \in [\langle {}^{\alpha}2, <_{\text{lex}}\rangle]^{\tau}$, there is a unique $s \in {}^{<\alpha}2$ of minimal length which is τ-splitting for A.

Proof of claim: First note that given any $s <_{\text{lex}} t$ both τ-splitting for A, $s \cap t$ is also τ-splitting for A. This is because $A^0_{s\cap t} \supseteq A^0_s$, and $A^1_{s\cap t} \supseteq A^1_t$, and A^0_s, A^1_t each embed τ by assumption. So given that there are any τ-splitting nodes for A, there is a unique one of minimal length; it remains to show that there are any such nodes at all.

Consider the relation $\sim_\tau$ defined on A by $x \sim_\tau y$ iff $A \cap [x,y]$ does not embed τ. Note that this is an equivalence relation, as if $x<y<z$ and $A \cap [x,z]$ does embed τ, then it also embeds e.g. $\tau+\tau+\tau$ and so at least one or the other of $A\cap[x,y]$ or $A \cap [y,z]$ must embed τ. Further, it is a condensation, i.e. the equivalence classes are intervals of A, and as such the ordering on A induces an ordering on these equivalence classes. We claim that this is a dense order; given $[x]<[y]$, by definition the interval between x and y in A embeds τ; but since $\tau+\tau\le\tau$, we have that $\tau+1+\tau\le\tau$; fix a copy of $\tau+1+\tau$ between x and y in A, and let z be the element of it corresponding to the 1 (we remark that this z is not necessarily unique, but this is not a problem); then $x \not\sim_\tau z$ and $z \not\sim_\tau y$ so $[x]<[z]<[y]$.

Now consider the set

$$S := \{s \in {}^{<\alpha}2 : \exists\, x, y \in A \text{ with } [x]<[y],$$
$$s = x\cap y, \text{ and } [x],[y] \text{ not extremal in } A/\sim_\tau\},$$

where here by *extremal* we mean maximal or minimal. Note that if s,t are both in S then so is $s\cap t$. It follows that the element of S of minimal length is unique; call this s_A. We claim that s_A is τ-splitting for A. Let $x<y$ witness that $s_A \in S$, so $s_A = x\cap y$ and $[x]<[y]$. Then since $[x]$ and $[y]$ are not extremal, there exist $x', y' \in A$ with $[x']<[x]$ and $[y]<[y']$, and $[x'],[y']$ also not extremal. Then by minimality $s_A = x'\cap y'$ also. But now, both x and x' extend $s_A^\frown\langle 0\rangle$ and both y and y' extend $s_A^\frown\langle 1\rangle$; in particular, $A^0_{s_A} \supseteq A\cap[x',x]$ and $A^1_{s_A} \supseteq A \cap [y,y']$; but since $[x']<[x]<[y]<[y']$, we have in particular that $x' \not\sim_\tau x$ and $y \not\sim_\tau y'$, so $A\cap[x',x]$ and $A\cap[y,y']$ both embed τ. $\blacksquare_{\text{Claim 1}}$

We now build an injection $f_A : {}^{<\omega}2 \to {}^{<\alpha}2$ which preserves both the tree structure and the lexicographic ordering of ${}^{<\omega}2$ by means of the following recursion:

$$f_A(\emptyset) = s_A, \text{ and for } i \in \{0,1\}, \text{ given } f_A(t) \text{ for some } t \in {}^{<\omega}2,$$
$$f_A(t^\frown\langle i\rangle) \text{ is the minimal-height } \tau\text{-splitting node for } A^i_{f_A(t)}.$$

Then in particular $f_A(t^\frown\langle i\rangle)$ extends $f_A(t)^\frown\langle i\rangle$, so f_A preserves both the tree structure and the lexicographic ordering of ${}^{<\omega}2$, as claimed.

Now we define a colouring $F : [\langle {}^{\alpha}2, <_{\text{lex}}\rangle]^\tau \to 2$ by, for $A \in [\langle {}^{\alpha}2, <_{\text{lex}}\rangle]^\tau$,

$$F(A) = \begin{cases} 0 & \text{if } h(f_A(\langle 0\rangle)) \geq h(f_A(\langle 1\rangle)); \\ 1 & \text{if } h(f_A(\langle 0\rangle)) < h(f_A(\langle 1\rangle)). \end{cases}$$

Claim 2. For any $s, t \in {}^{<\alpha}2$ which are both τ-splitting for A and have $s <_{\text{lex}} t$, there is $B \in [A]^\tau$ with $f_B(\langle 0\rangle) = s$, $f_B(\langle 1\rangle) = t$, and $f_B(\emptyset) = s \cap t$.

Proof of claim: By definition each of A^0_s, A^1_s, A^0_t, and A^1_t embed τ, and since s and t do not extend each other, all four of these sets are disjoint. Since $\tau + \tau + \tau + \tau \leq \tau$, we can find some $\tau_0, \tau_1, \tau_2, \tau_3$, all bi-embeddable with τ, such that $\tau = \tau_0 + \tau_1 + \tau_2 + \tau_3$. Then let B be formed of the disjoint union of a copy of τ_0 in A^0_s, a copy of τ_1 in A^1_s, a copy of τ_2 in A^0_t, and a copy of τ_3 in A^1_t.

Since all of B extends $s \cap t$ and $B^0_{s\cap t}$, $B^1_{s\cap t}$ both embed τ (as $s \cap t^\frown\langle 0\rangle \sqsubseteq s$ and $s \cap t^\frown\langle 1\rangle \sqsubseteq t$), it follows that $s \cap t$ is the minimal τ-splitting node for B. Then since every element of $B^0_{s\cap t}$ extends s and s is τ-splitting for B by construction, $f_B(\langle 0\rangle) = s$, and similarly $f_B(\langle 1\rangle) = t$. $\blacksquare_{\text{Claim 2}}$

Now, using Claim 2, we will show that no $A \in [\langle {}^{\alpha}2, <_{\text{lex}}\rangle]^\tau$ can be homogeneous for the colouring F defined above. First observe that if A is such that $h(f_A(\langle 0\rangle)) = h(f_A(\langle 1\rangle))$, then, applying Claim 2 with e.g. $s = f_A(\langle 0\rangle)$, $t = f_A(\langle 11\rangle)$, we obtain some $B \in [A]^\tau$ with $h(f_B(\langle 0\rangle)) < h(f_B(\langle 1\rangle))$, and so $F(A) \neq F(B)$.

It follows that for some $A \in [\langle {}^{\alpha}2, <_{\text{lex}}\rangle]^\tau$ to be homogeneous for F, it must be the case either that $h(f_B(\langle 0\rangle)) > h(f_B(\langle 1\rangle))$ for all $B \in [A]^\tau$, or that $h(f_B(\langle 0\rangle)) < h(f_B(\langle 1\rangle))$ for all $B \in [A]^\tau$. But now, for any $s <_{\text{lex}} t$ in ${}^{<\omega}2$, we can apply Claim 2 to $f_A(s)$ and $f_A(t)$, and obtain either that for every $s <_{\text{lex}} t \in {}^{<\omega}2$, $h(f_A(s)) > h(f_A(t))$, or that for every $s <_{\text{lex}} t \in {}^{<\omega}2$, $h(f_A(s)) < h(f_A(t))$. Both situations are impossible, as ${}^{<\omega}2$ contains both ω-sequences and ω^*-sequences in $<_{\text{lex}}$, so in either case we would get an infinite descending sequence of ordinals. $\square$