Colouring copies of the rationals and the Kinna-Wagner principle

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Joint work with Jonathan Schilhan

Kinna-Wagner

Definition: The (original) Kinna-Wagner principle

KWP_1 is the following statement:

$$\forall X\ \exists \alpha \in \mathrm{Ord} \text{ such that } X \text{ injects into } \mathcal{P}(\alpha).$$

This is a choice principle which is strictly weaker than AC (KWP_0). It implies that every set can be linearly ordered. Our base theory throughout the talk will be ZF.

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The partition relation symbol

Notation: Copies of τ in $\langle L, <\rangle$

For $\langle L, <\rangle$ a linear order, τ an order type, write $[\langle L, <\rangle]^\tau$ for the set of subsets of L ordered as τ in the induced suborder.

Definition: Partition relation symbol

For $\langle L, <\rangle$ a linear order and σ, τ order types,

$$\langle L, <\rangle \to (\sigma)^\tau$$

is the statement that for any $F : [\langle L, <\rangle]^\tau \to 2$, thought of as a *colouring* of the copies of τ in $\langle L, <\rangle$, there is some $H \in [\langle L, <\rangle]^\sigma$ which is *homogeneous* or *monochromatic* for F, in the sense that $|F\text{ " }[H]^\tau| = 1$.

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Background

Minimal relations on generalisations of the reals

In Udine, worked on relations of the form

$$\langle {}^{\alpha}2, <_{\mathrm{lex}} \rangle \to (\tau)^{\tau},$$

where α is an ordinal, with Thilo Weinert and Jonathan Schilhan.

The behaviour of these relations can change quite a lot as α increases, but some order types τ are so self-similar that a relation $\langle {}^{\alpha}2, <_{\mathrm{lex}} \rangle \to (\tau)^{\tau}$ can never hold:

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Let τ be an order type with $\tau + \tau \leq \tau$. Then for all ordinals α,

$$\langle {}^{\alpha}2, <_{\mathrm{lex}} \rangle \not\to (\tau)^{\tau}.$$

Examples of τ satisfying this condition: the order type of the rationals η, the order type of the real line λ, any countable non-scattered order.

Proof sketch

Fix τ, α. We build a colouring $c : [\langle {}^{\alpha}2, <_{\mathrm{lex}} \rangle]^{\tau} \to 2$ with no homogeneous set in the following way: for $A \in [\langle {}^{\alpha}2, <_{\mathrm{lex}} \rangle]^{\tau}$, we associate two nodes $s_A^0, s_A^1 \in {}^{<\alpha}2$ with A, and colour according to which is longer.

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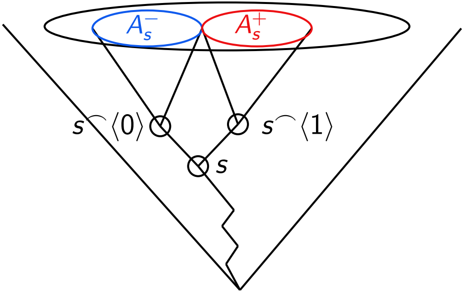
τ-splitting nodes

$$A \in [\langle {}^{\alpha}2, <_{\mathrm{lex}} \rangle]^{\tau}$$



Say s is *τ-splitting for A* if both A_s^- and A_s^+ contain copies of τ, where

$$A_s^- := \{x \in A : s^\frown\langle 0 \rangle \sqsubseteq x\};$$
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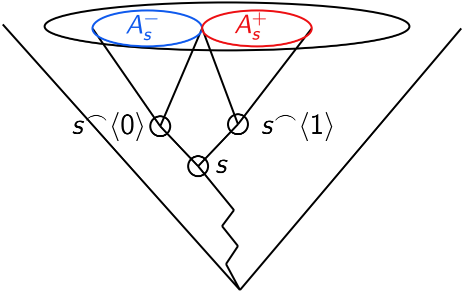

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Proof sketch of Prop. 1 ctd.

Claim 1

For any $A \in [\langle {}^{\alpha}2, <_{\mathrm{lex}} \rangle]^{\tau}$,

1. there exist $s \in {}^{<\alpha}2$ which are τ-splitting for A;
2. given any $s, t \in {}^{<\alpha}2$ which are both τ-splitting for A, so is their maximal common initial segment.

It follows that there is a unique τ-splitting node of minimal length; call this s_A.

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The nodes s_A^0 and s_A^1

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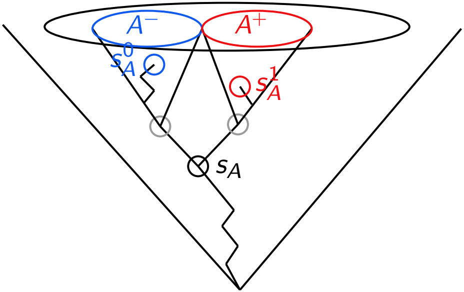

Write s_A^0 for the unique minimal-length τ-splitting node of A extending $s_A {}^\frown \langle 0 \rangle$, and s_A^1 for the unique minimal-length τ-splitting node of A extending $s_A {}^\frown \langle 1 \rangle$.

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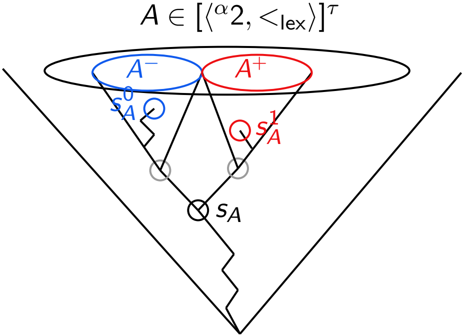

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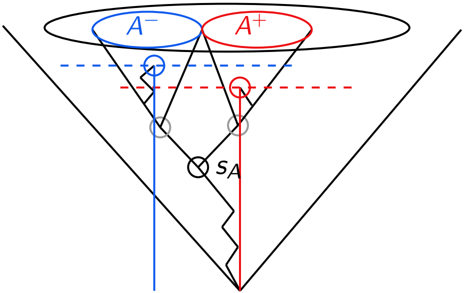

We colour according to which of $\text{len}(s_A^-)$, $\text{len}(s_A^+)$ is longer, i.e.

$$c(A) = \begin{cases} 0 & \text{if } \text{len}(s_A^-) \geq \text{len}(s_A^+); \\ 1 & \text{if } \text{len}(s_A^-) < \text{len}(s_A^+). \end{cases}$$

Proof sketch of Prop. 1 ctd.

Claim 2

If $t_0 <_{\text{lex}} t_1 \in {}^{<\alpha}2$ are any τ-splitting nodes for A, then there is some $A' \in [A]^\tau$ with $s^0_{A'} = t_0$ and $s^1_{A'} = t_1$.

Claim 3

There is an embedding of the tree $\langle {}^{<\omega}2, \subseteq \rangle$ in the τ-splitting nodes of A, preserving both the tree structure and the lexicographic ordering.

These two claims together give that our colouring c can have no homogeneous set, as ${}^{<\omega}2$ contains both ω-sequences and ω^*-sequences in $<_{\text{lex}}$, so a homogeneous set in either colour would induce an infinite descending sequence of ordinals. $\square$

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The question

Result (Schilhan)

It is consistent with ZF that there is a linear order $\langle L, < \rangle$ with

$$\langle L, < \rangle \to (\eta)^\eta.$$

The linear order $\langle L, < \rangle$ in the model Schilhan built has the property that the set L consists of sets of reals, prompting him to ask the following question:

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If $\langle L, < \rangle$ is a linear order with $\langle L, < \rangle \to (\eta)^\eta$, is it possible that $L \subseteq \mathbb{R}$, or more generally $L \subseteq \mathcal{P}(\alpha)$ for α some ordinal?

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The answer

Proposition 2

Let $\langle L, <\rangle$ be a linear order such that the set L injects into the power set of an ordinal. Then

$$\langle L, <\rangle \not\to (\eta)^{\eta}.$$

Equivalently, if $\langle L, <\rangle$ is such that $\langle L, <\rangle \to (\eta)^{\eta}$, then the set L witnesses a failure of the Kinna-Wagner principle KWP_1.

We remark that if the *order* $\langle L, <\rangle$ embedded in some $\langle {}^{\alpha}2, <_{\mathrm{lex}}\rangle$, Proposition 1 would immediately gives $\langle L, <\rangle \not\to (\eta)^{\eta}$, but the set L injecting into some ${}^{\alpha}2$ does not imply the existence of such an order-embedding.

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The colouring

Proof sketch

We prove the contrapositive. Fix $\langle L, <\rangle$ a linear order, α an ordinal, and $i : L \hookrightarrow {}^{\alpha}2$ an injection; we build a colouring witnessing

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going by a series of five cases. For $A \in [\langle L, <\rangle]^{\eta}$, the colour $c(A)$ will be determined by the properties of its image $i \text{ " } A \subseteq {}^{\alpha}2$ and the properties of the images of $A' \in [A]^{\eta}$.

First, we split into cases according to the order-theoretic properties of $\{i \text{ " } A' : A' \in [A]^{\eta}\}$, considered as sub-orders of $\langle {}^{\alpha}2, <_{\mathrm{lex}}\rangle$; then we split into further cases according to the the topological properties of the $i \text{ " } A'$.

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Case 1: There are $A' \in [A]^\eta$, $A'' \in [A]^\eta$ such that i " A' is ordered as η and i " A'' is not ordered as η in $\langle {}^\alpha 2, <_{lex} \rangle$.

Simply order according to whether i " A is ordered as η or not:

$$c(A) = \begin{cases} 0 & \text{if } i \text{ " } A \text{ is ordered as } \eta \text{ in } \langle {}^\alpha 2, <_{\mathrm{lex}} \rangle; \\ 1 & \text{otherwise.} \end{cases}$$

Idea: if we remain in this case we can always change colour by reducing to a subset whose image has a different order type.

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Apply the colouring used in Proposition 1 to i " A (the argument that there is no homogeneous set is more involved).

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Cases 1 and 2 deal with the situation where some $A' \in [A]^\eta$ has $i \,"\, A'$ ordered as η in $\langle {}^\alpha 2, <_{\text{lex}} \rangle$, so for the rest of the proof we can assume that this is never the case.

We now consider the restrictions of the images $i \,"\, A'$ to ${}^\beta 2$, where $\beta \leq \alpha$ is the minimal ordinal sufficient to distinguish the members of $i \,"\, A$.

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In this case, i " A is non-scattered (bi-embeddable with η) and we can colour it with the colouring from Proposition 1. Again, the argument that there is no homogeneous set is more involved.

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Disregard A unless i " A has two points which are "more isolated" than all others. Ask whether $\langle L, <\rangle$ and $\langle {}^\alpha 2, <_{\text{lex}}\rangle$ agree on the ordering of these two points and colour according to that.
Lots of WLOGing: we can assume that for any $x, y \in A$ there is some $A' \in [A]^\eta$ such that x and y are the two points picked out in this way. Now, since i " A is not ordered as η, there are $x_0 < y_0$ with $i(x_0) <_{\text{lex}} i(y_0)$ *and* $x_1 < y_1$ with $i(x_0) >_{\text{lex}} i(y_0)$.

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Flowchart for determining $c(A)$

Properties of i " A' for $A' \in [A]^\eta$

(3-5: Never ordered as η)

1: Some ordered as η, some not ordered as η; colour based on this

3: Some with isolated points, some with no isolated points; colour based on this

5: Always has isolated pts; Lots of WLOGing. Look at two most isolated points and colour based on whether the $\langle L, <\rangle$-order and the $\langle {}^\alpha 2, <_{\text{lex}}\rangle$-order agree on these points

2: All ordered as η; Prop. 1 colouring*

4: Never has isolated pts; Prop. 1 colouring†