Determinacy of Infinite Borel Games

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April 13, 2026

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Infinite Games

We consider two player infinite games of perfect information. The games will be played between Players `I` and `II` on game trees as defined below:

- A **position** is a finite sequence of moves. A position f **continues** a position g if f can be obtained by adding one move to g.
- A set of positions T with $\varnothing \in T$ is called a **game tree** if for any position $f \in T$ there is some $g \in T$ that continues it, and every initial segment of g is in T.
- A **play** in T is an infinite sequence whose every finite segment belongs to T. The set of all plays in T is denoted by $[T]$.
- A **game** is the pair (T, A) where $A \subseteq [T]$ is called the **winning set** for Player `I`.
- To play the game, each player selects a strategy and plays according to it. A game is determined if one of the players has a winning strategy.

Borel Hierarchy

For a position $f \in T$ let T_f be the smallest subtree of T containing f. We define a topology on $[T]$ by setting $[T_f]$ to be the basic open sets. This gives us open and closed sets in $[T]$ - we say they are Borel Σ_1 and Borel Π_1 respectively.
Building up from there, we have the following:

$$\text{Level } \alpha \begin{cases} \text{A set } A \text{ is } \Sigma_\alpha \text{ if } A = \bigcup_{i \in \mathbb{N}} A_i \\ \text{for some some } A_i \text{ from } \Pi_{\alpha_i} \text{ where } \alpha_i < \alpha \text{ for } i = 1, 2, \ldots \\ \\ \text{A set } A \text{ is } \Pi_\alpha \text{ if } A = \bigcap_{i \in \mathbb{N}} A_i \\ \text{for some some } A_i \text{ from } \Sigma_{\alpha_i} \text{ where } \alpha_i < \alpha \text{ for } i = 1, 2, \ldots \end{cases}$$

We say that a game (T, A) is Σ_α (resp. Π_α) if A is Σ_α (resp. Π_α) in the topology on $[T]$. Such games are also called Borel games.

What is needed for Borel Determinacy?

In Second Orded Arithmetic (i.e. natural numbers and their subsets):

- Gale-Stewart Theorem shows open and closed games are determined
- Further results extend this up to Σ_3 and Π_3.

Paris showed $ZF \vdash \Sigma_4$ determinacy, but was ZF necessary?
Surprisingly, the answer is yes! Friedman showed that if we only have up to Σ_α replacement, we cannot have $\Sigma_{\alpha+5}$ determinacy, while Martin refined the result to $\Sigma_{\alpha+4}$ (and even $\Sigma_{\alpha+3}$ for infinite α)

General Idea

Let ZFC^- be ZFC without the Powerset.
First find β such that $L_\beta \models ZFC^-$, and consider games on $\omega^{<\omega}$.
Suppose that `I` only wins if her moves constitute a set of sentences true in L_β. If such a strategy was in L_β, then L_β would encode the set of sentences true in it - which is impossible!
By carefully modifying the win condition, we can find a Σ_4 game that is not determined in L_β, thus proving that $ZFC^- \not\models \Sigma_4$ determinacy.

Proof of Borel Determinacy

Firstly, say that a game G' covers a game G if winning strategies in G' give rise to winning strategies in G.
Let T be a tree. We will inductively find an open cover of (T, A) for each Borel $A \subseteq [T]$.
Let A be Σ_α i.e. $A = \bigcup_n A_n$. Let the game G_n be closed and a cover for (T, A_n) (by induction hypothesis). The main idea of finding the cover is to have moves $2n$ and $2n+1$ completely decide the set A_n.

Proof of Borel Determinacy

Move $2n$ (I)	Choose a I-imposed subgame of G_n and play a move in it. Call this subgame G_n^{I}
Move $2n+1$ (II)	Either Choose a II-imposed subgame of G_n^{I} all of whose plays are in A_n OR Choose a position in G_n^{I} none of whose plays are in A_n

Player I wins iff player II chooses the first option at any point of the game, thus this game is indeed open and hence determined. The bulk of the work is then to show that winning strategies in this game (for both I and II) give rise to the winning strategies for (T, A)

Thank You!