Part III Essay:

Generic and Virtual Large Cardinals

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32. Generic and Virtual Large Cardinals .... Professor B. Löwe

Generic large cardinals were introduced in Foreman's 1998 invited talk at the *International Congress of Mathematicians* in Berlin: a *generic large cardinal axiom* is the statement that a large cardinal exists in a generic extension [2]. These axioms are generally equiconsistent to the corresponding large cardinal axiom.

Virtual large cardinals were introduced by Gitman and Schindler as a generalisation of Schindler's notion of *remarkable cardinals* (equivalently, "virtually supercompact") [3]. Typically, the consistency strength of a virtual large cardinal is considerably lower than that of the corresponding large cardinal notion. Virtual versions of extendible, huge, and I3 cardinals are discussed in [3]; virtual versions of superstrong, Woodin, and Berkeley cardinals are discussed in [1].

An essay written under this title will give a general introduction into generic and virtual large cardinals and then analyse one particular large cardinal notion and its generic and virtual forms in detail.

Section 1: Introduction

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Large Cardinal Axioms

- Asserts the existence of cardinals with some properties
- Cannot be proved within ZFC
- Motivation: Add new axioms to decide statements above ZFC
- First few candidates: "Stronger axioms of infinity", which generalize different properties of

Smaller large cardinal axioms

- strong limit and regular -> inaccessible cardinals

Definition 1.1.1. *(Inaccessible) A cardinal κ is* **inaccessible** *if it is a regular strong limit*[1]*.*

- satisfies compactness theorem -> weakly compact cardinals

Definition 1.1.3. *(Weakly compact) A cardinal κ is* **weakly compact** *if for every $\mathcal{L}_{\kappa,\kappa}$-language*[2] *$L$ with a set of non-logical symbols S of cardinality $|S| \leqslant \kappa$, and every set Φ of L-sentences, the following holds: if Φ is κ-satisfiable, then Φ is satisfiable.*

- satisfies Ramsey's theorem -> Ramsey cardinals

Definition 1.1.8. *(Ramsey) A cardinal* κ *is* **Ramsey** *if* $\kappa \to (\kappa)^{<\omega}$.

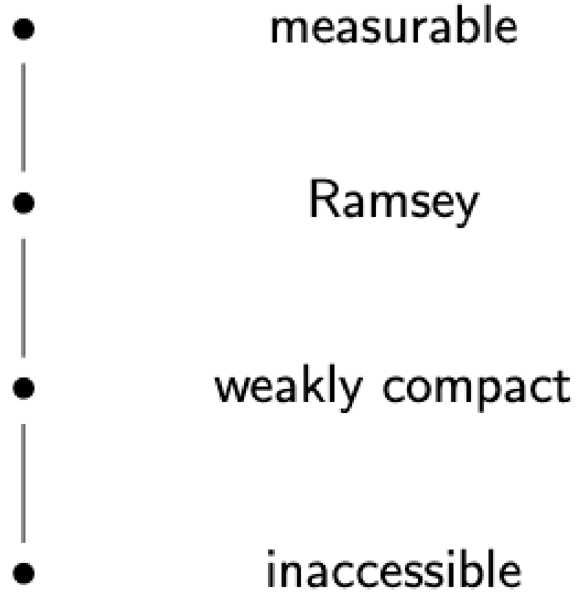
- carries a nonprincipal ultrafilter -> Measurable cardinals

Definition 1.1.10. *(Measurable) A cardinal* κ *is* **measurable** *if there is a function* $\mu : \mathcal{P}(\kappa) \to \{0,1\}$ *that is*

(i) nontrivial: $\mu(\{x\}) = 0 \quad \forall x \in \kappa$;

(ii) κ*-additive: for any* $\lambda < \kappa$, *for any family* $\{A_\alpha; \alpha < \lambda\}$ *of size* λ *of disjoint subsets of* κ, *have* $\mu(\bigcup_{\alpha<\lambda} A_\alpha) = \sum_{\alpha<\lambda} \mu(A_\alpha)$;

(iii) obeys $\mu(\kappa) = 1$ *and* $\mu(\varnothing) = 0$.



Consistency strength hierarchy:

- Cons:={Cons(ZFC+A); A finite set of sentences}
- For any theory T, C(T):= consequences of T
- Cons(A)Cons(B), if C(A)Cons C(B)Cons

How to get above measurable?

Scott’s & Keisler’s Results

- Equivalent characterizations of being measurable:
  - Existence of -complete ultrafilter on .
  - Existence of elementary embedding j: V M with M transitive class and crit(j) = .

- Large cardinal notions above measurables:
  - Require the existence of elementary embedding(s)
  - Strengthened by e.g. requiring more closure properties of M

Elementary embedding characterizations

Class-sized embeddings are of the forms:

1. There exists an elementary embedding , , transitive (and has some additional properties)
2. For every , there exists an elementary embedding , , transitive, (j and M have some additional properties depending on).

Examples:

Definition 1.1.12. *(Measurable, elementary embedding characterization) A cardinal* κ *is* **measurable** *if there is a transitive class* $M \subset V$ *and an elementary embedding* $j : V \to M$ *with* $crit(j) = \kappa$*.*

And some examples of larger large cardinal notions are:

Definition 1.1.13. *(Strong) A cardinal* κ *is* **strong** *if for every ordinal* $\lambda > \kappa$ *there exists a transitive class* $M \subset V$ *and an elementary embedding* $j : V \to M$*, such that* $V_\lambda \subset M$*,* $j(\kappa) > \lambda$*, and* $crit(j) = \kappa$*.*

Definition 1.1.14. *(Supercompact) A cardinal* κ *is* **supercompact** *if for every ordinal* $\lambda > \kappa$ *there exists a transitive class* $M \subset V$ *and an elementary embedding* $j : V \to M$*, such that* $M^\lambda \subset M$ *(i.e. every function* $\lambda \to M$ *is an element of* M*),* $j(\kappa) > \lambda$*, and* $crit(j) = \kappa$*.*

Definition 1.1.15. *(*n*-huge) A cardinal* κ *is* **n-huge** *(*$n \in \omega$*) if there exists a transitive class* $M \subset V$ *and an elementary embedding* $j : V \to M$*, such that* $M^{j^n(\kappa)} \subset M$ *and* $crit(j) = \kappa$*.*

Elementary embedding characterizations

Set-sized embeddings are of the forms:

1. For every there is an elementary embedding with (and some additional properties depending on).
2. For some there is an elementary embedding with (and some additional properties).

Examples:

Definition 1.1.16. *(Extendible) A cardinal κ is* **extendible** *if for every $\lambda > \kappa$ there is an elementary embedding $j : V_\lambda \to V_{\beta_\lambda}$ with $crit(j) = \kappa$ and $j(\kappa) > \lambda$.*

Definition 1.1.17. *(Rank-into-rank) A cardinal κ is* **rank-into-rank** *if for some $\lambda > \kappa$ there is an elementary embedding $j : V_\lambda \to V_\lambda$ with $crit(j) = \kappa$.*

A problem remains

- Large cardinals not affected by “small” forcings:

Theorem 2.1.7. *(Lévy-Solovay) Let κ be a measurable cardinal in the ground model. Let $(P,<)$ be a notion of forcing such that $|P| < \kappa$. Then κ is measurable in the generic extension.*

- CH affected by small forcings, e.g.

(This was unsatisfactory)

- So large cardinals still cannot decide CH

- Relax conditions on the elementary embedding characterizations-> **Generic & Virtual Large Cardinals**

Generic large cardinals (Foreman 1998)

Idea: Elementary embeddings and target models **definable** in a **forcing extension**

(We focus on large cardinal notions characterized by class-sized embeddings)

- Given a large cardinal notion characterized by: Exists (with some properties)
- A cardinal is **generically** : In some exists (…)

- Given a large cardinal notion characterized by: For every , exists   (with some properties depending on)
- A cardinal is **generically** : For every , in some exists (…)

Generic large cardinals (Foreman 1998)

- Examples

Definition 1.2.1. *(generically measurable) A cardinal κ is* **generically measurable** *if in a forcing extension $V[G]$ there is a transitive class $M \subset V[G]$ and an elementary embedding $j : V \to M$ with $crit(j) = \kappa$.*

Definition 1.2.2. *(generically strong) A cardinal κ is* **generically strong** *if for every $\lambda > \kappa$, in a forcing extension $V[G]$ there is a transitive class $M \subset V[G]$ and an elementary embedding $j : V \to M$, such that $V_\lambda \subset M$, $crit(j) = \kappa$, and $j(\kappa) > \lambda$.*

Definition 1.2.3. *(generically supercompact) A cardinal κ is* **generically supercompact** *if for every $\lambda > \kappa$, in a forcing extension $V[G]$ there is a transitive class $M \subset V[G]$ and an elementary embedding $j : V \to M$ with $crit(j) = \kappa$ and $j''\lambda \in M$.*[6]

Definition 1.2.4. *(generically n-huge) A cardinal κ is* **generically n-huge** *if in a forcing extension $V[G]$ there is a transitive class $M \subset V[G]$ and an elementary embedding $j : V \to M$ with $crit(j) = \kappa$ and $(M^{j^n(\kappa)})^{V[G]} \subset M$.*

Compare with:

Definition 1.1.12. *(Measurable, elementary embedding characterization) A cardinal κ is* **measurable** *if there is a transitive class $M \subset V$ and an elementary embedding $j : V \to M$ with $crit(j) = \kappa$.*

Virtual large cardinals (Gitman 2018)

Idea:

1. Elementary embeddings are **definable** in a **forcing extension**
2. Elementary embeddings are **set-sized**, with **target models contained in V**

(First look at large cardinal notions characterized by class-sized embeddings)

- Given a large cardinal notion characterized by:
    1. Exists (with some properties); or
    2. For every exists   (…)
- A cardinal is **virtually** : For every , in some exists (…)

Note: it is equivalent to take

Virtual large cardinals (Gitman 2018)

- Examples (virtual forms of large cardinals characterized by class-sized embeddings)

Definition 1.2.6. *(Virtually measurable) A cardinal κ is* **virtually measurable** *if for every $\lambda > \kappa$, in a forcing extension $V[G]$ there is a transitive class $M \subset V$ and an elementary embedding $j : V_\lambda \to M$, such that $crit(j) = \kappa$.*

Definition 1.2.7. *(Virtually strong) A cardinal κ is* **virtually strong** *if for every $\lambda > \kappa$, in a forcing extension $V[G]$ there is a transitive class $M \subset V$ and an elementary embedding $j : V_\lambda \to M$, such that $V_\lambda \subset M$, $crit(j) = \kappa$, and $j(\kappa) > \lambda$.*

Compare with:

Definition 1.1.12. *(Measurable, elementary embedding characterization) A cardinal κ is* **measurable** *if there is a transitive class $M \subset V$ and an elementary embedding $j : V \to M$ with $crit(j) = \kappa$.*

Virtual large cardinals (Gitman 2018)

Idea:

1. Elementary embeddings are **definable** in a **forcing extension**
2. Elementary embeddings are **set-sized**, with **target models contained in V**

(Now look at large cardinal notions characterized by set-sized embeddings)

- Given a large cardinal notion characterized by:
  1. For every exists   (with some properties depending on); or
  2. For some exists   (with some properties)
- A cardinal is **virtually** : The embeddings exists in some forcing extensions
  1. For every , in some there is (…)
  2. For some , in some there is (…)

Virtual large cardinals (Gitman 2018)

- Examples (virtual forms of large cardinals characterized by set-sized embeddings)

Definition 1.2.8. *(Virtually extendible) A cardinal κ is* **virtually extendible** *if for every $\lambda > \kappa$, in a forcing extension $V[G_\lambda]$ there is an elementary embedding $j_\lambda : V_\lambda \to V_{\beta_\lambda}$, such that $crit(j_\lambda) = \kappa$ and $j_\lambda(\kappa) > \lambda$.*

Definition 1.2.9. *(Virtually n-huge*) A cardinal κ is* **virtually n-huge*** *if for some $\lambda > \kappa$, in a forcing extension there is an elementary embedding $j : V_\lambda \to V_\beta$ with $crit(j) = \kappa$ and $j^n(\kappa) < \lambda$.*

Definition 1.2.10. *(Virtually rank-into-rank) A cardinal κ is* **virtually rank-into-rank** *if for some $\lambda > \kappa$, in a forcing extension there is an elementary embedding $j : V_\lambda \to V_\lambda$ with $crit(j) = \kappa$.*

Compare with:

Definition 1.1.16. *(Extendible) A cardinal κ is* **extendible** *if for every $\lambda > \kappa$ there is an elementary embedding $j : V_\lambda \to V_{\beta_\lambda}$ with $crit(j) = \kappa$ and $j(\kappa) > \lambda$.*

Definition 1.1.17. *(Rank-into-rank) A cardinal κ is* **rank-into-rank** *if for some $\lambda > \kappa$ there is an elementary embedding $j : V_\lambda \to V_\lambda$ with $crit(j) = \kappa$.*

Section 2: Measurability

A natural dividing line between "smaller" large cardinals and "larger" large cardinals.

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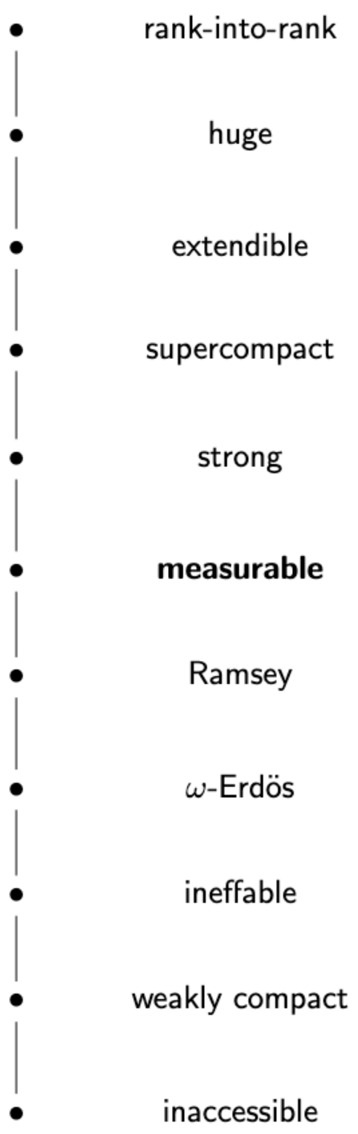


2.3 Virtually measurable cardinals
(Bounds on consistency strength & Consistency with L)

Definition 2.3.1. *(Ineffable) A cardinal κ is* **ineffable** *if, given any sequence $(A_\alpha)_{\alpha<\kappa}$ of sets such that $A_\alpha \subset \alpha$ for all $\alpha < \kappa$, there exists an $A \subset \kappa$ such that $\{\alpha \in \kappa; A \cap \alpha = A_\alpha\}$ is stationary.*

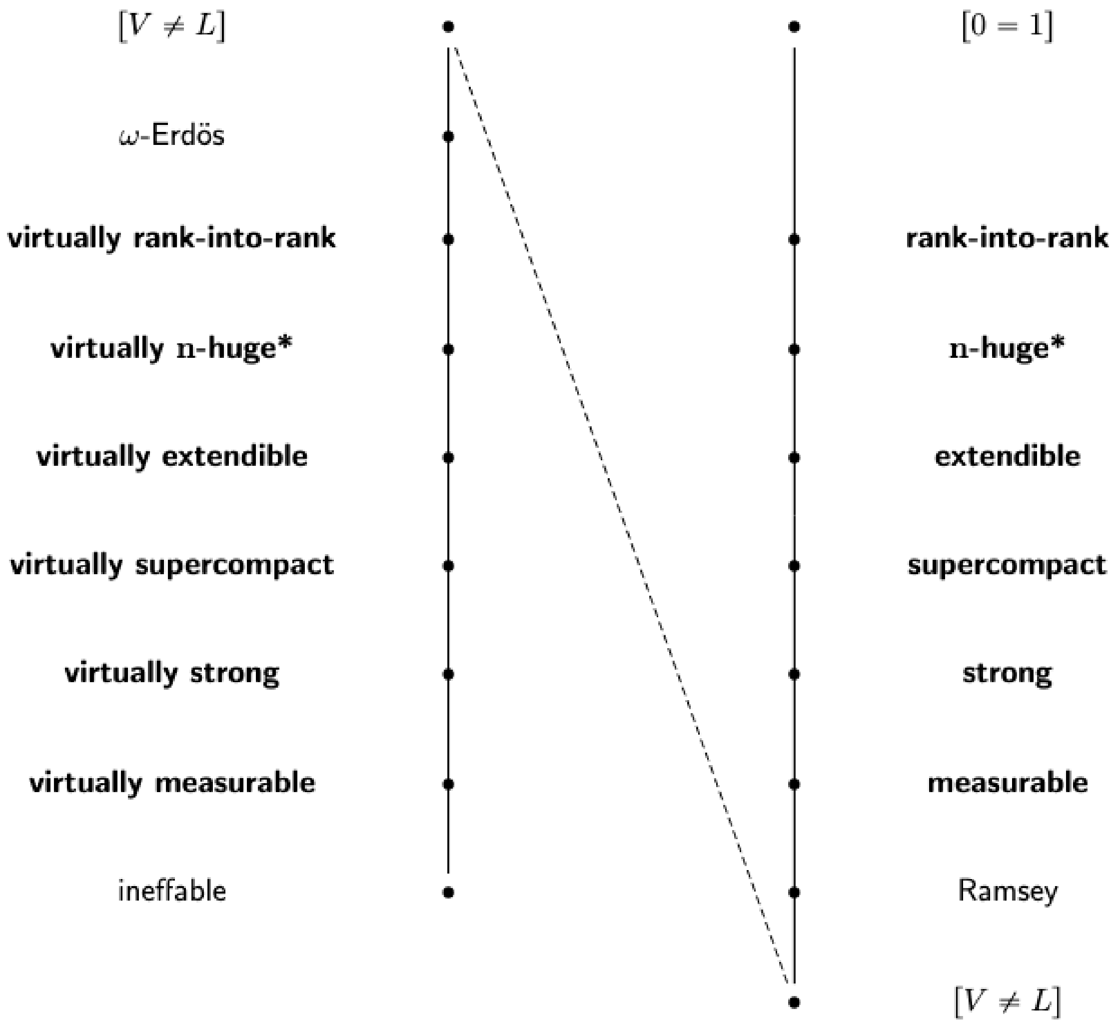
Theorem 2.3.2. *Let κ be virtually measurable, then κ is ineffable.*

Definition 2.3.3. *(Erdös cardinals) For every limit ordinal α, the* **Erdös** *cardinal η_α is the least κ such that*

$$\kappa \to (\alpha)_2^{<\omega}.$$

$$Cons(\omega\text{-Erdös}) \overset{(1)}{\geqslant} Cons(\text{(v)rank-into-rank}) \geqslant Cons(\text{(v)}n\text{-huge*}) \overset{(2)}{\geqslant} Cons(\text{(v)extendible}) \geqslant Cons(vMC)$$

Theorem 2.3.11. *If κ is ω-Erdös, then $L \models$ "κ is ω-Erdös".*



- Virtual large cardinals form a hierarchy mirroring their traditional counterparts
- They are all consistent with L
- Can look at virtual versions of familiar large cardinal notions instead of other combinatorial principals

2.2 Generically measurable cardinals
(Some details on " can be generically measurable")

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1. Start with a transitive model M of ZFC, in which a measurable cardinal carries some -complete ideal containing all singletons:

   - Forcing poset P = {X ; X }
   - G generic over M, then G is a M-ultrafilter on extending the dual filter of .
   - Inside M[G] form the generic ultrapower N:= (M)
   - Elementary embedding with crit(j) = .

2. A –complete ideal on is called precipitous if N is well-founded.
   - Mostowski collapse gives generic measurability on

3. Theorem: if is measurable, then there is a forcing extension in which and carries a precipitous ideal.
    - Force over M by Coll(,<) so that becomes in M[G].
    - In M[G] form the ideal on generated by the dual of G.
    - Show that for some S   in M[G], the ideal | = {X } is precipitous.

4. Conclude:
    - M’:= M[G], where M ZFC + MC.
    - M’ “ carries a precipitous ideal”.
    - Generic ultrapower construction in M’[G’] gives elementary embedding with crit(j) = and N’ transitive.
    - M’ ZFC + gMC().

Thanks!

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