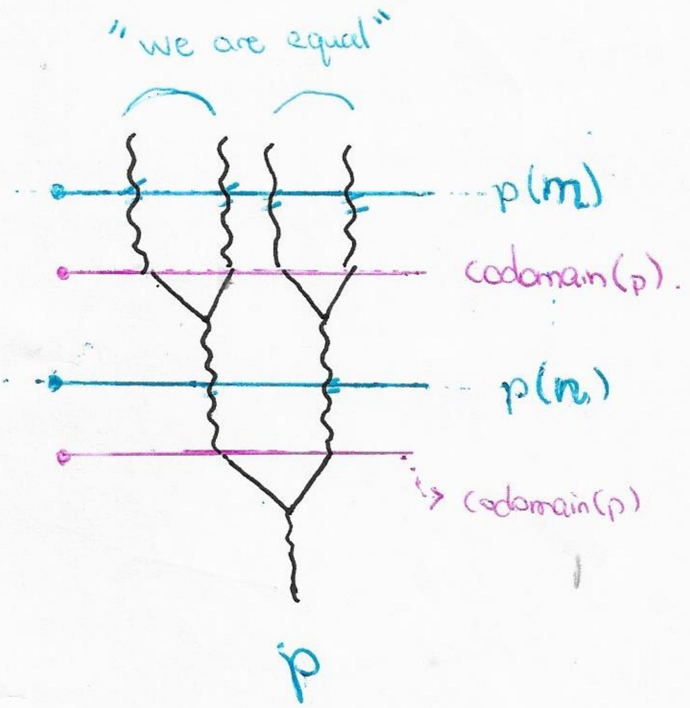
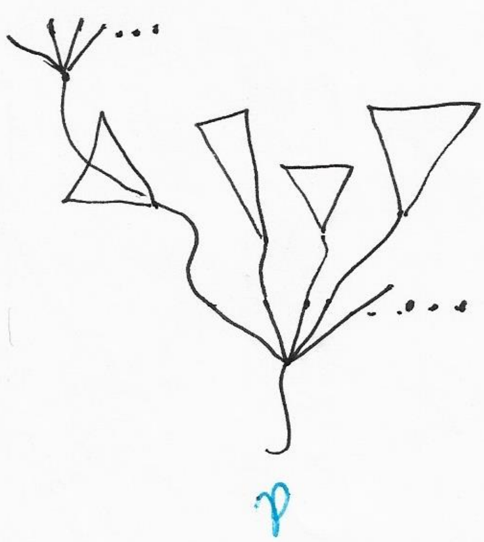
Silver vs. Laver.

Silver tree:



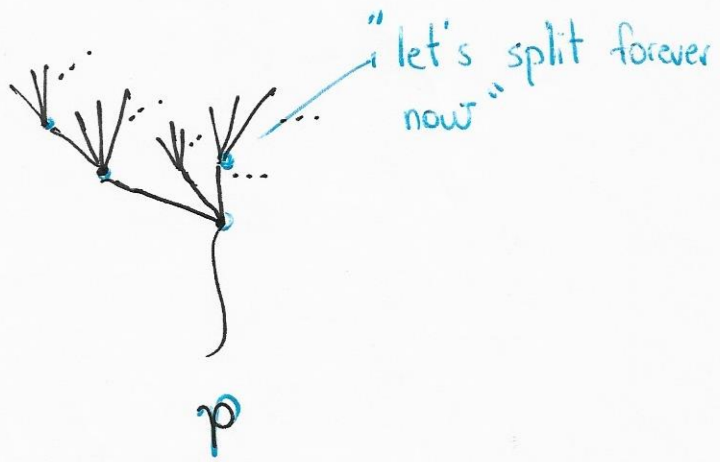
- $\varphi : \omega \to 2$, finite, and $|\omega \setminus \mathrm{dom}(p)| = \omega$;
- one may assign values on non-splitting levels;
- splitting levels represent the codomain of the partial function;

Miller tree:



- $p \subseteq \omega^{<\omega}$ is a perfect tree;
- every splitting node has infinitely many successors.

Laver tree:



- $p \subseteq \omega^{<\omega}$ is a Miller tree;
- Above the stem... every node is infinitely splitting.

Tree ideals

- $p^0 = \{ A \mid \forall p \in \mathbb{P}\ \exists q \leq p\ ([q] \cap A = \emptyset)\}.$

$\mathbb{V}$	—	v^0
$\mathbb{M}$	—	m^0
$\mathbb{L}$	—	ℓ^0

Note: p^0 is a σ-ideal.

Let us prove

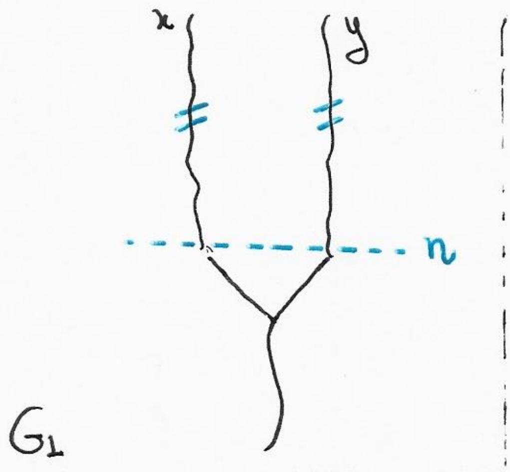
$$\mathrm{cov}(v^0) < \mathrm{cov}(\ell^0)$$

in the Laver model.

Why graphs?

Consider the graph

$$G_\perp = \{(x,y)\in (2^\omega)^2 \mid \exists!\, n\in\omega\ (x(n)\neq y(n))\}.$$



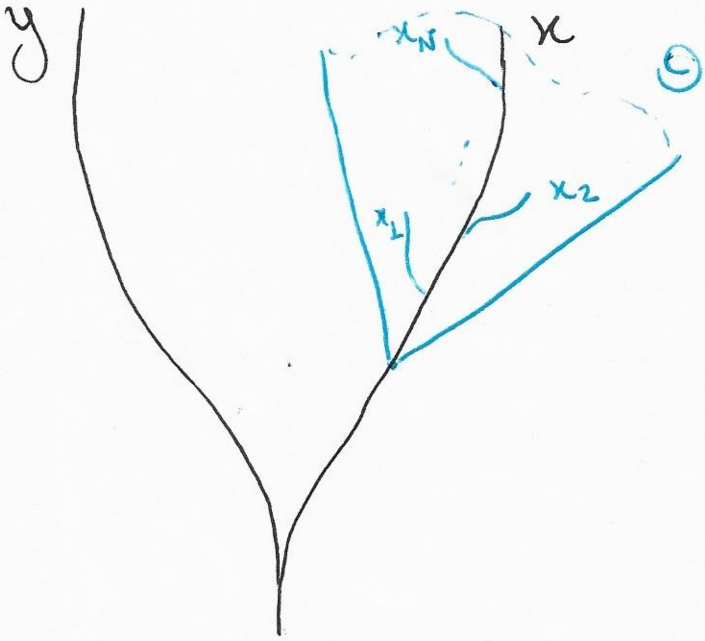
- Let $I_{G_\perp}$ be the σ-ideal generated by Borel G-independent sets.
- The forcing notion $\mathrm{Bor}(2^\omega)\setminus I_{G_\perp}$ is equivalent to $\mathbb{V}$.

- As an immediate consequence, $\mathbb{V}$ increases $\mathrm{cov}(I_{G_\perp})$.
- Silver-quasi reals are exactly the reals avoiding Borel sets of $I_{G_\perp}\cap V$.

Fact (Zapletal). $p\longmapsto [p]$ is a dense embedding from $\mathbb{V}$ to $\mathrm{Bor}(2^\omega)\setminus I_{G_\perp}$.

In fact: If $A\subseteq 2^\omega$ is an analytic set, either $A\in I_{G_\perp}$ or $[p]\subseteq A$, for some $p\in\mathbb{V}$.

Graphs of Finite Local Degree (FLD)



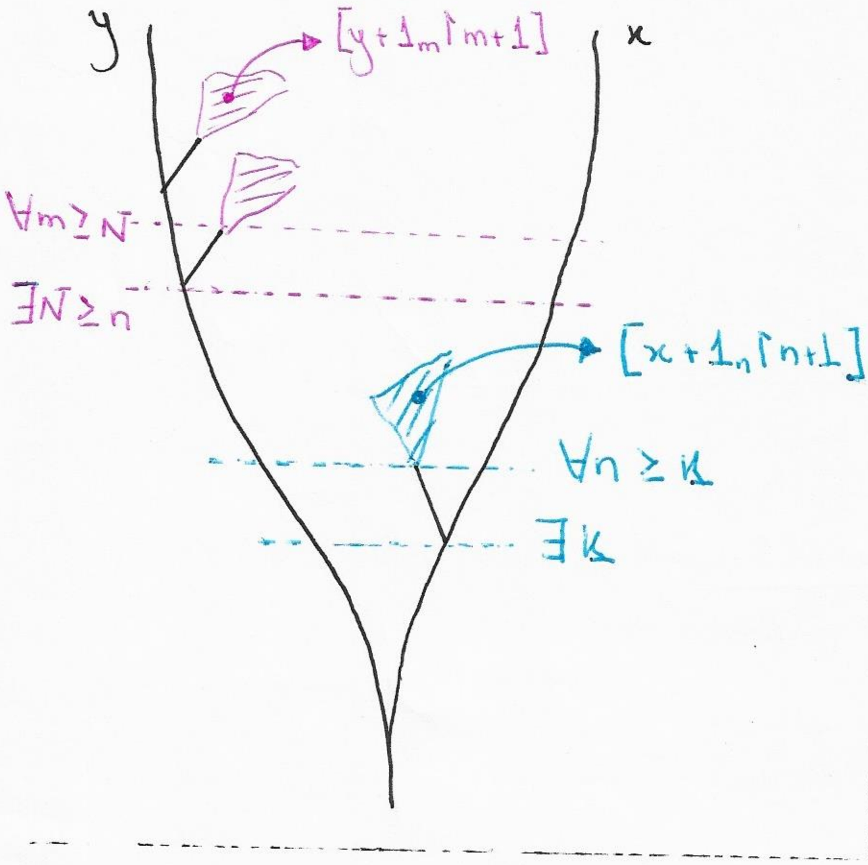
- X is Polish & $G \subseteq X^2$ is a graph;
- $(x,y) \in G$;

Then $\exists \mathcal{O} \ni x,\ y \notin \mathcal{O}$: $(\mathcal{O} \times \{y\}) \cap G$ is finite.

Fact. FLD graphs are locally countable.

Our Example: G_1 is FLD

Lemma. For FLD graphs:



$\forall (x,y) \in G$:

$\exists K\ \forall n \geq K\ \exists N \geq n\ \forall m \geq N$:

$([x+1_n \restriction n+1] \times [y+1_m \restriction m+1]) \cap G = \emptyset$.

! no edges between blue and pink stuff.

!! the proof requires compactness.

Guiding reals

Let p be either a Miller or a Laver tree.

For $\sigma \in p$ splitting node, if $(\sigma_n)_{n\in\omega}$ is an enumeration of $succ_p(\sigma)$; assume:

- $p * \sigma_n$ decides $\dot{x}\restriction n$; This can be done using the pure decision property.
- $(x_n)_{n\in\omega}$ is a sequence with $x_n \in [T_{p*\sigma_n}(\dot{x})]$, $\forall n \in \omega$;
- Since 2^ω is compact, one may also assume (x_n) is convergent and we let $x_\sigma \doteq \lim x_n$.

Note. x_σ does not depend on the choice of (x_n) and x_σ is ground-model.

Define x_σ as the σ-guiding real of p. Assume every node has a defined guiding real.

Inducing edges.

Miller: $q \leq p$ induces G-edges on $\dot{x}$ iff:

$$\forall y \in [q] \left((f(y), x_{st(q)}) \in G \right).$$

Laver: $q \leq p$ induces G-edges on $\dot{x}$ iff:

$$\forall \sigma \in succ_q(st(q)) \left((z_\sigma, z_{st(q)}) \in G \right).$$

In any case:

$$D_G(\dot{x}) \doteq \{ q \leq p \mid q \text{ induces } G\text{-edges on } \dot{x} \}.$$

Claim. For both Miller and Laver:

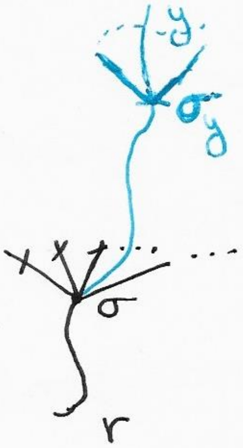
If $D_G(\dot{x})$ is <u>not</u> dense below p, then there exists $q \leq p$ such that

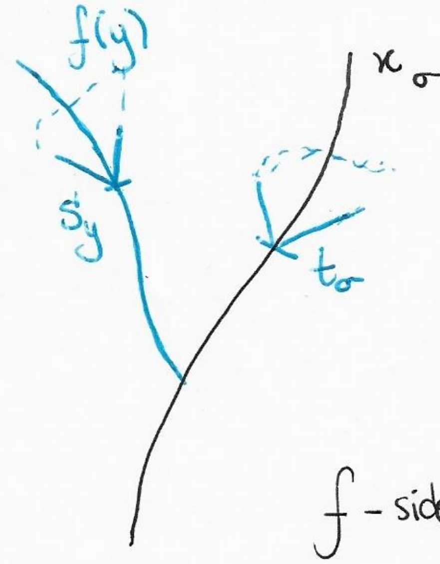
$$f''[q] \text{ is } G\text{-independent},$$

whenever G is <u>closed</u>.

Note: The Miller notion of inducing edges is much stronger than the Laver notion.

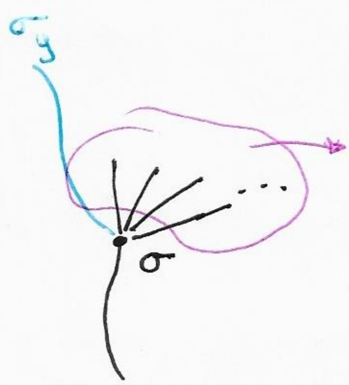
The case with Miller :





f - side.

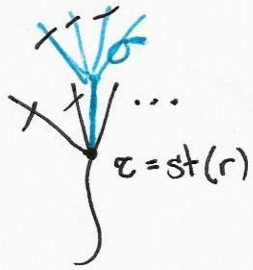
- r is a condition, witnessing that $D_\sigma(\dot{x})$ is not dense below p ;
- $y \in [r]$ is such that $(f(y), x_\sigma) \notin G$.
- $s_y \subseteq f(y)$, $t_\sigma \subseteq x_\sigma$, and
$$([s_y] \times [t_\sigma]) \cap G = \emptyset$$
- By continuity : $\exists\, \sigma_y \supseteq \sigma : f''[\sigma_y] \subseteq s_y$.
- Prune the tree leaving only σ_y onwards and repeat the argument for the remaining tree — i.e., without σ_y :



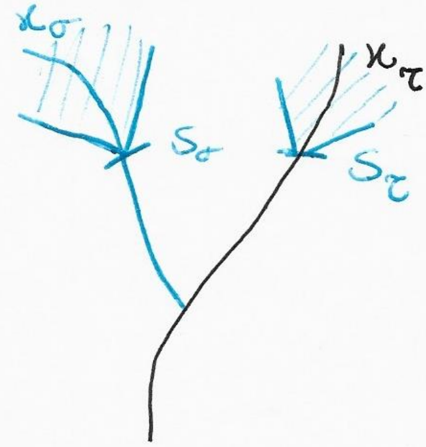
consider this tree, and repeat!

- Construct q as a fusion!

The case with Laver.

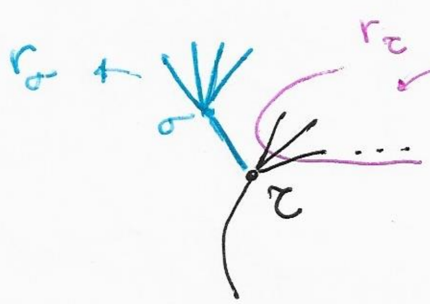


r side



f side

- $r \le q$ witness $D_G(\dot{x})$ is not dense below p.
- $\sigma \in succ_r(\tau)$ is such that $(x_\sigma, x_\tau) \notin G$;
- $s_\sigma \subseteq x_\sigma$, $s_\tau \subseteq x_\tau$,
  $$([s_\sigma] \times [s_\tau]) \cap G = \emptyset;$$
- There are:
  - $r_\tau \le_0 r$; $r_\tau \Vdash s_\tau \subseteq \dot{x}$;
  - $r_\sigma \le_0 r * \sigma$: $r_\sigma \Vdash s_\sigma \subseteq \dot{x}$.
- Use a simple fusion argument:



repeat the argument for this tree:

In case of Miller:

- If G is locally countable, then $D_G(\dot{x})$ is not dense below p.

In case of Laver:

- G closed FLD <u>may not</u> imply that $D_G(\dot{x})$ is dense below p!
- We aim to show that our goal can still be reached:

  $\exists q \leq_0 p : f''[q]$ is a Borel G-independent;

Assume $\dot{x}$ is not ground-model.

Define a rank on p:

For $\sigma \in p$,

$rk_D(\sigma) = 0 \longleftrightarrow \exists q \leq_0 p * \sigma : q \in D$,

where

$D = \{ q \leq p \mid \forall \sigma \in succ_q(st(q)) \, (x_\sigma \neq x_{st(q)}) \}$.

If $rk_D(\sigma) \neq 0$, then

$rk_D(\sigma) = \min \{ \alpha \mid \exists U \in [\omega]^\omega \, \forall n \in U \, (rk_D(\sigma^\frown n) < rk_D(\sigma)) \}$

<u>Note</u>: rk_D is defined on p because $\dot{x}$ is not ground-model!

Note also: $D = D_G(x)$, where G is the complete graph on 2^ω.

(See e.g. Lemma 7.3.31 of Bartoszyński–Judah book for more rank arguments!)

Where frontiers come in handy:

- $A \subseteq p$ is a frontier iff it is
  - a maximal antichain, and
  - $\forall x \in [p]\, \exists!\, n \in \omega : x \restriction n \in A$.

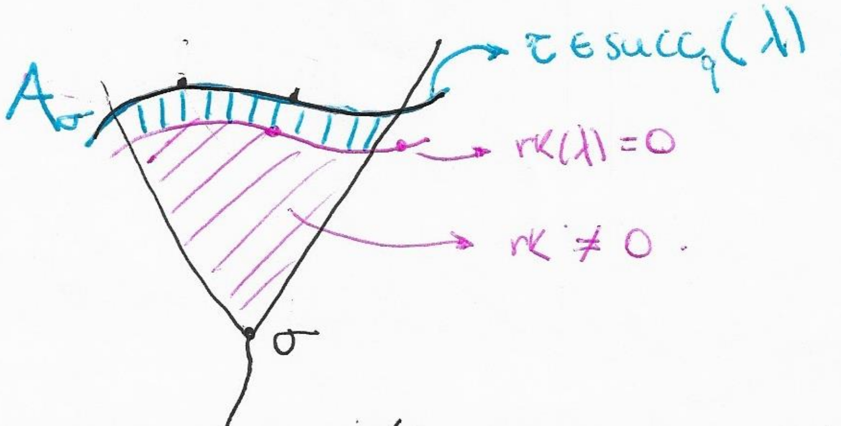
Lemma.

$\exists q \leq_0 p\ \forall \sigma \in q$ of $rk \neq 0$ $\exists A_\sigma \subseteq q * \sigma$ frontier: $\forall \tau \in A_\sigma$:

(1) $\tau \in succ_q(\lambda)$ & $rk(\lambda) = 0$;

(2) $x_\tau = x_\sigma$; and

(3) any proper splitting node of $q * \lambda$ has non-zero rank.



(See Gro87, or BKW16)

Note also: $\mathbb{D} = \mathbb{D}_G(\dot{x})$, where G is the complete graph on 2^ω.

(see e.g. Lemma 7.3.31 of Bartoszyński - Judah book for more rank-type arguments)

Where frontiers come in handy.

- $A \subseteq p$ is a frontier iff it is:
  - a maximal antichain, and
  - $\forall x \in [p]\, \exists!\, n \in \omega : x \restriction n \in A$.

Lemma.

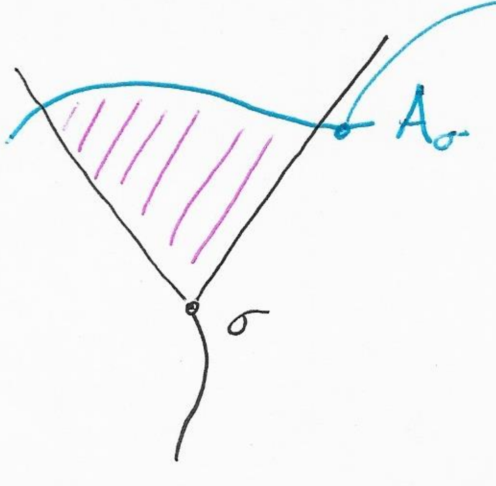
$$\exists q \leq_0 p\ \forall \sigma \in q\ \exists A_\sigma \subseteq q * \sigma \text{ frontier}: \forall \tau \in A_\sigma:$$

(1) $rk(\tau) = 0$;

(2) $\forall \lambda \in succ_q(\tau)$, $x_\lambda \neq x_\sigma$, & $\underline{x_\tau = x_\sigma}$

(3) any proper splitting subnode of τ in q has non-zero rank.

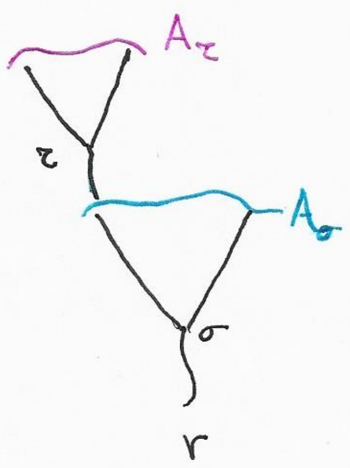
(see Groszek 87, or Brendle-Khomskii-Wohofsky 2016)



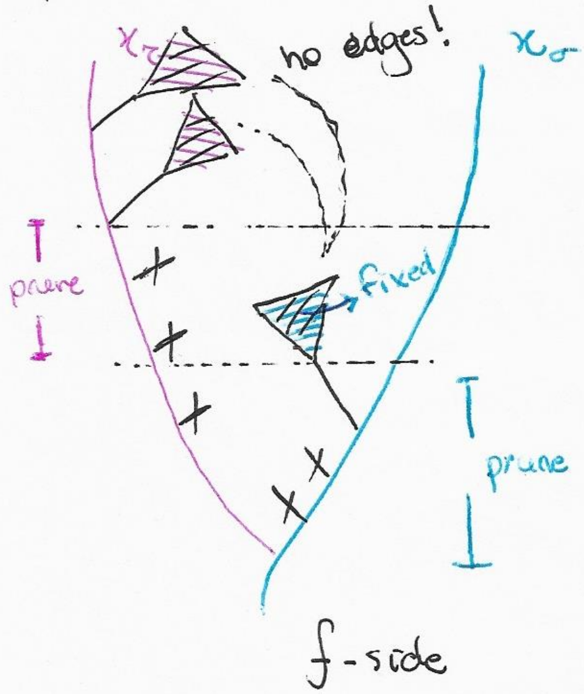
$rk(\tau) = 0$ determines a unique $x_\sigma = $ "x_{A_σ}".

Let $r \leq p$ and $\sigma \in \mathrm{split}(p)$. Let $A_\sigma \subseteq r * \sigma$ be a frontier as in the previous lemma.

For $\tau \in \mathrm{succ}_{r*\sigma}(A_\sigma)$, also let $A_\tau \subseteq r * \tau$ be as in the previous lemma.



- Now we have to use the lemma on page 4, which holds for closed FLD graphs, for the reals x_σ and x_τ.

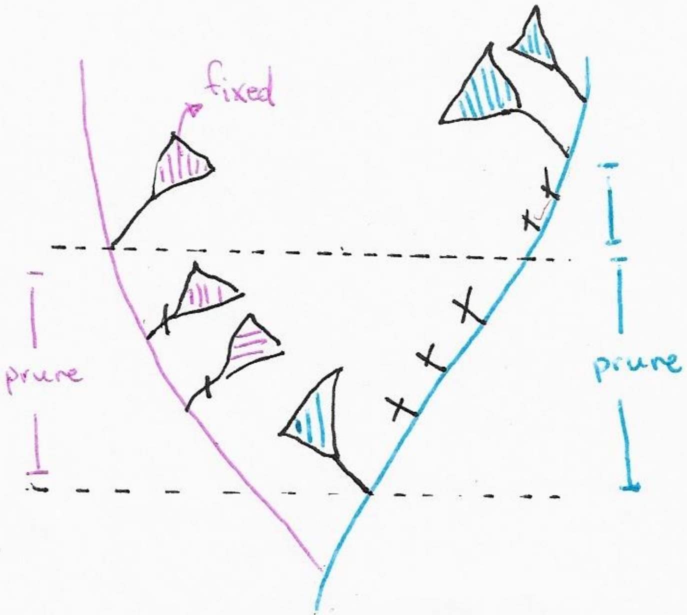


- After not much pruning (kill finitely many successors) on the successors of both A_σ and A_τ, for a fixed $\tau' \neq \tau$; $\tau' \in \mathrm{succ}_r A_\sigma$;

$$(f''[r * \tau'] \times f''[r * \lambda]) \cap G = \emptyset \, ;$$

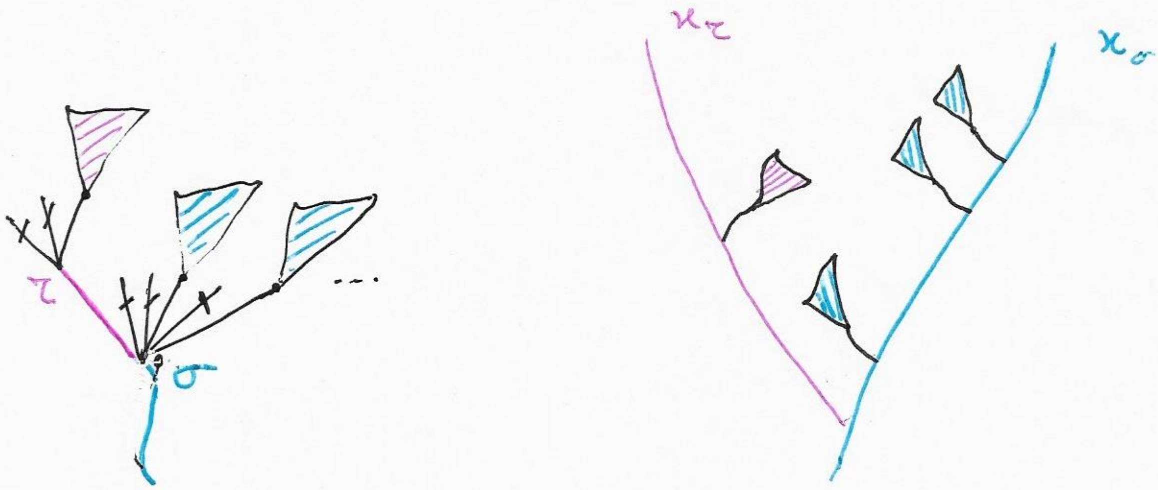
for $\lambda \in \mathrm{succ}_r(A_\tau)$, as in the picture.

- Now we do some "back-and-forth":



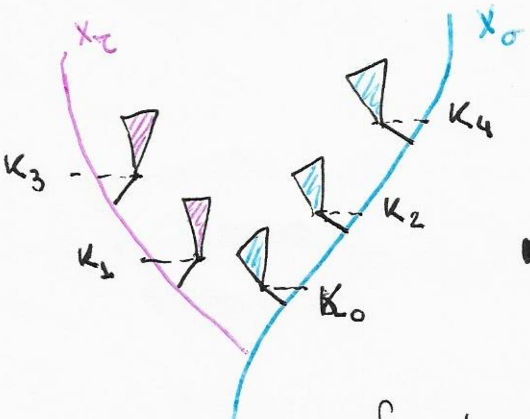
There are no edges between pink and blue open sets!

- We fixed now a node on the pink side, and apply the lemma again. We are left with:



depicting of the process.

- In the end of this process we have:

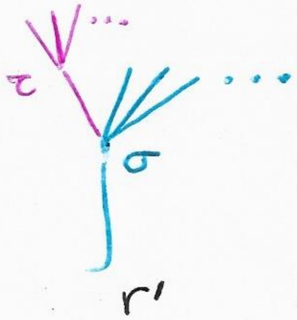


f-side.

- (k_n) strictly increasing;

- $(f''[r * \tau'] \times f''[r * \lambda]) \cap G = \emptyset$.

Once we have a picture like this:

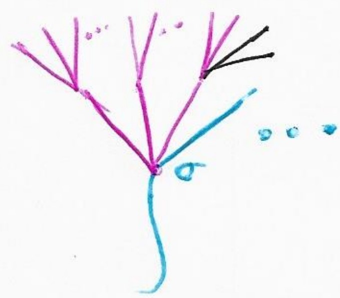


For $\tau' \neq \tau$, $\tau' \in succ(\sigma)$

$\lambda \in succ(\tau)$ &

$(f''[r' * \tau'] \times [r' * \lambda]) \cap G = \emptyset$,

we proceed and fix another node – i.e:



- When everything is pink, we finish the first two levels / frontiers and "go up".

Remark: There are some technicalities here that were overlooked. See slides to fill out the gaps.

- Through the described fusion argument we obtain $q \leq_0 p$ such $f''[q]$ is Borel, since f is 1-1 on q, and G-independent!