My Part III Essay Plans

Large Cardinals: Characterizations of Weakly Compact Cardinals

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What is a Part III Essay?

- **Part III** is a 9 month taught masters course.

> ***Content of a Part III Essay***
>
> *“The object of a typical essay is to give an exposition of a piece of mathematics which is scattered over several books or papers.”*
>
> — Part III Essay Booklet: Guidelines and Titles 2024-25

- “Essay title”: Large Cardinals
- My essay topic: Characterizations of Weakly Compact Cardinals
- Due date: Thursday, May 08, 2025, at 12:00

General essay plan

Three main parts for my essay:

A. Define characterizations of weakly compact cardinals
B. Explain/show which characterizations imply inaccessibility
C. Give implication proofs between characterizations (all will end up being equivalent)

Preliminaries

Definition (Cofinality)

The cofinality of a cardinal κ, denoted $\mathrm{cf}(\kappa)$, is the cardinality of the least set $S \subset \kappa$ such that $\bigcup S = \kappa$.

Definition (Regular cardinal)

A cardinal κ is regular if $\mathrm{cf}(\kappa) = \kappa$.

Definition (Strong limit)

A cardinal κ is a strong limit if for all $\lambda < \kappa$, $2^\lambda < \kappa$.

Definition (Inaccessible cardinal)

For κ an uncountable cardinal:

- κ is weakly inaccessible if it is regular and a limit.
- κ is (strongly) inaccessible if it is regular and a strong limit.

Remark

$\mathrm{ZFC} \nvDash \mathrm{Con}(\mathrm{ZFC}) \to \mathrm{Con}(\mathrm{ZFC} + \text{“there exists an inaccessible cardinal”})$

List of characterizations

The following are equivalent, given κ an inaccessible cardinal. Weakly compact cardinals are normally defined by either `WCT` or `PP`.

1. (`WCT`) The infinitary language $\mathcal{L}_{\kappa,\kappa}$ satisfies the Weak Compactness Theorem.
2. (`WCT2`) The infinitary language $\mathcal{L}_{\kappa,\omega}$ satisfies the Weak Compactness Theorem.
3. (`PP`) The partition relation $\kappa \to (\kappa)^2_2$ holds.
4. (`PP2`) The partition relation $\kappa \to (\kappa)^n_\lambda$ holds for every $\lambda \in \kappa, n \in \omega$.
5. (`TP`) κ has the tree property.
6. (`EP`) κ has the (Keisler) Extension Property.
7. (`ID`) κ is Π^1_1-indescribable.
8. (`OP`) κ has the “long total order property”/κ is a “Hausdorff cardinal”.
9. (`FP`) κ has the “filter extension property”.

Definitions

Definitions: WCT, WCT2

Reminder (Language of first-order logic)

The language of first-order logic consists of:

- (Finitary) **non-logical symbols** (i.e. function and relation symbols), and associated arities.
- An infinite set of **variables** V.
- **Terms** defined recursively from non-logical symbols and variables.
- **Formulas** defined recursively from **atomic formulas** and logical connectives.

Definition (Infinitary logic $\mathcal{L}_{\kappa,\lambda}$)

The language of $\mathcal{L}_{\kappa,\lambda}$ is formed by adding **two new rules** for creating new formulas:

- **Infinitary conjunctions and disjunctions**: $\bigwedge_{\alpha<\mu}$, $\bigvee_{\alpha<\mu}$, for $\mu < \kappa$.
- **Infinitary quantification**: $\exists_{\alpha<\mu}, \forall_{\alpha<\mu}$, for $\mu < \lambda$.

Definitions: WCT, WCT2

Remark

$\mathcal{L}_{\omega,\omega}$ is the usual language of first-order logic.

Definition (μ-satisfiable)

A set of sentences Σ is μ-satisfiable iff every subset of Σ of cardinality less than μ is satisfiable.

Compactness Theorem for $\mathcal{L}_{\omega,\omega}$

Whenever Σ is a set of sentences,

$$\Sigma \text{ is } \omega\text{-satisfiable} \leftrightarrow \Sigma \text{ is satisfiable.}$$

Definition (Weak Compactness Theorem)

For $\kappa \geqslant \lambda$, $\mathcal{L}_{\kappa,\lambda}$ satisfies the Weak Compactness Theorem iff whenever Σ is a set of sentences *using at most κ non-logical symbols*,

$$\Sigma \text{ is } \kappa\text{-satisfiable} \leftrightarrow \Sigma \text{ is satisfiable.}$$

Definitions: WCT, WCT2

Definition (Weak Compactness Theorem)

$\mathcal{L}_{\kappa,\lambda}$, $\kappa \geqslant \lambda$ satisfies the Weak Compactness Theorem iff whenever Σ is a set of sentences, *using at most κ non-logical symbols*,

$$\text{(Any} < \kappa\text{-sized subset of } \Sigma \text{ is satisfiable)} \Rightarrow \Sigma \text{ is satisfiable.}$$

(c.f. Compactness Theorem for first-order logic)

1. (`WCT`) The infinitary language $\mathcal{L}_{\kappa,\kappa}$ satisfies the Weak Compactness Theorem.
2. (`WCT2`) The infinitary language $\mathcal{L}_{\kappa,\omega}$ satisfies the Weak Compactness Theorem.

Definitions: PP, PP2

Ramsey's Theorem

For r, k positive naturals: Whenever the r-sets (i.e. r-sized subsets) of ω are k-coloured, then there is a monochromatic ω-set (i.e. an infinite set in which all r-sets are coloured the same).

Definition (Partition relation)

Let $\alpha, \beta, \gamma, \delta$ be cardinals. Then the *partition relation*

$$\beta \longrightarrow (\alpha)^{\gamma}_{\delta}$$

holds iff "whenever the γ-sets of β are δ-coloured, there is a monochromatic α-set".

Remark

Ramsey's Theorem: $\omega \longrightarrow (\omega)^{r}_{k}$ for all $r, k \in \omega$.

3. (PP) The partition relation $\kappa \to (\kappa)^2_2$ holds.
4. (PP2) The partition relation $\kappa \to (\kappa)^n_\lambda$ holds for every $\lambda \in \kappa$, $n \in \omega$.

Definitions: TP

Definition (Trees)

- A *tree* is a partially ordered set $(T, <_T)$ such that for any $t \in T$ the set $u \in T; u <_T T$ of $<_T$-predecessors of t is well-ordered by $<_T$.
- For the sake of simplicity all trees are assumed to have a minimal element, called the *root*.
- The *αth level* of a tree T consists of every $t \in T$ such that $\{x \in T; x <_T y\}$ has order type α.
- The *height* of T is the least α such that the αth level of T is empty.
- A *branch* of T is a maximal chain in T.
- A *cofinal branch* of T is a branch with elements at every non-empty level of T.

Definitions: TP

Definition (κ-trees)

A κ-*tree* is a tree of height κ, each of whose levels has cardinality less than κ.

Definition (κ-Aronszajn trees)

A κ-*Aronszajn* tree is a κ-tree with no cofinal branch.

Definition (The tree property)

Let κ be a cardinal.
κ has the *tree property* iff every κ-tree has a cofinal branch (i.e. there are no κ-Aronszajn trees).

5. (TP) κ has the Tree Property.

Definitions: EP

Definition (The Extension Property)

Let κ be a cardinal.

κ satisfies the *(Keisler) Extension Property* if for any $R \subset V_\kappa$, there is a transitive set $X \supsetneq V_\kappa$, and a subset $S \subset X$, such that $(V_\kappa, \in, R) \preceq (X, \in, S)$.

6. (EP) κ has the (Keisler) Extension Property.

Definitions: ID

Reminder (Variables in first order logic)

Let M be a structure, and let φ be a formula. Variables which appear in φ are interpreted as *elements of M*.

Definition (Language of higher-order logic)

In nth order predicate logic:

- **Variables each have an "order"**, from 1 to n. Our set of variables has **infinitely many variables of each order** from 1 to n.
- Quantifiers may be applied to variables of any order.
- For each pair of variables X, z, of orders $k+1$, k respectively, have a **new atomic formula** $X(z)$.
- Formulas are defined inductively from atomic formulas and logical connectives as usual.

Definitions: ID

In nth order (finitary) logic:

- **Variables each have an "order"**, from 1 to n. Our set of variables has **infinitely many variables of each order** from 1 to n.
- Quantifiers may be applied to variables of any order.
- For each pair of variables X, z, of orders $k+1, k$ respectively, have a **new atomic formula** $X(z)$.
- Formulas are defined inductively from atomic formulas and logical connectives as usual.

Definition (Interpretation in higher-order logic)

In nth order logic:

- nth order variables are interpreted as elements of $\mathbb{P}^{n-1}(M)$.
- The new atomic formula $X(z)$ is interpreted as "$z \in X$".

Definitions: ID

Reminder (Σ_n, Π_n formulas in first-order logic)

Let $n > 0$ be a natural number, and let φ be a formula.

- φ is Σ_0 or Π_0 iff it is quantifier-free, i.e. Δ_0.
- φ is Σ_n (resp. Π_n) if it is of the form $\exists x_0. \cdots \exists x_k. \psi(x_0, \ldots, x_k)$ (resp. $\forall x_0. \cdots \forall x_k. \psi(x_0, \ldots, x_k)$), where ψ is a Π_{n-1} (resp. Σ_{n-1}) formula, $x_0, \ldots, x_k$ are variables.

Remark

Complexity is measured in terms of first-order quantification.

Definition (Σ^n_m, Π^n_m formulas)

"Measure complexity in terms of $(n+1)$th order quantification".
Let $m > 0$ be a natural number, and let φ be a formula.

- φ is Σ^n_0 or Π^n_0 iff all its quantified variables are of order at most n.
- φ is Σ^n_m iff it is of the form $\exists X_0. \cdots \exists X_k. \psi(X_0, \ldots, X_k)$, where ψ is a Π^n_{m-1} formula, and $X_0, \ldots, X_k$ are variables of order $(n+1)$.
- Similarly define Π^n_m.

Definitions: ID

Definition (Indescribable cardinals)

A cardinal κ is Π^n_m-indescribable if whenever $U \subset V_\kappa$ and φ is a Π^n_m sentence such that $(V_\kappa, \in, U) \vDash \varphi$, then for some $\alpha < \kappa$, $(V_\alpha, \in, U \cap V_\alpha) \vDash \varphi$.

Similarly can define Σ^n_m-indescribability.

Remark

Motto: "Can't describe with a Π^n_m formula how big κ is".

7. (ID) κ is Π^1_1-indescribable.

Definitions: 0P

Definition (Long total order property)

A cardinal κ satisfies the *long total order property* if for any total order of cardinality κ, there is some sub-order which is a strictly monotone sequence (i.e. well-order or reverse of well-order), of order type κ.

8. (0P) κ has the "long total order property".

Definitions: FP

Definition (Filter)

A *filter* on a set S is a set F of subsets of S such that:

- F is **closed under intersections**: If $A, B \in F$, then $A \cap B \in F$.
- F is **closed under supersets**: If $A \in F$, $B \supset A$, then $B \in F$.
- F is **non-empty, non-trivial**: $\emptyset \notin F$, $S \in F$.

An *ultrafilter* is a maximal filter on S (wrt inclusion).

Definition (Algebra of sets)

An *algebra over a set* S is a collection G of subsets of S such that:

- G is **closed under intersections**: If $A, B \in G$, then $A \cap B \in G$.
- G is **closed under complements**: If $A \in G$, then $S \setminus A \in G$.
- F is **non-empty**: $S \in G$

Definitions: FP

Definition (κ-complete)

A filter or algebra X is κ-complete if it is closed under interesections of size $< \kappa$.
That is, for any $\lambda < \kappa$, $\{X_\alpha;\, \alpha \in \lambda\} \subset X$, we have $\bigcap_{\alpha\in\lambda} X_\alpha \in X$.

Remark

A σ-algebra is exactly a ω-complete algebra.

Definition (Filter extension property)

A cardinal κ satisfies the "filter extension property" iff for any algebra of sets B over κ, with $|B| = \kappa$, any κ-complete filter over B is contained in a κ-complete ultrafilter over B.

9. (FP) κ has the "filter extension property".

Which characterizations imply inaccessibility?

Which characterizations imply inaccessibility?

Assuming κ is uncountable, (1-4, 6, 7, 8) imply inaccessibility:

1. (WCT) The infinitary language $\mathcal{L}_{\kappa,\kappa}$ satisfies the Weak Compactness Theorem.
2. (WCT2) The infinitary language $\mathcal{L}_{\kappa,\omega}$ satisfies the Weak Compactness Theorem.
3. (PP) The partition relation $\kappa \to (\kappa)^2_2$ holds.
4. (PP2) The partition relation $\kappa \to (\kappa)^n_\lambda$ holds for every $\lambda \in \kappa, n \in \omega$.
5. (TP) κ has the Tree Property. [Independent of ZFC*]
6. (EP) κ has the (Keisler) Extension Property.
7. (ID) κ is Π^1_1-indescribable.
8. (OP) κ has the "long total order property".
9. (FP) κ has the "filter extension property". [I don't know]

*Assuming weakly compact cardinals are consistent.

$\mathrm{TP}(\kappa) \to \mathtt{IC}(\kappa)$?

Reminder (The tree property)

κ has the *tree property* iff every κ-tree has a cofinal branch, i.e. there are no κ-Aronszajn trees.

Facts

- $\mathrm{TP}(\kappa)$ implies "κ is regular".
- ZFC proves $\aleph_1$ does not have the tree property.
- $\mathrm{TP}(\aleph_2)$ is independent of ZFC:
  - CH implies $\neg\mathrm{TP}(\aleph_2)$.
  - Mitchell (1972) showed that $\mathrm{TP}(\aleph_2)$ is equiconsistent to the existence of weakly compact cardinals.
  - As a consequence, if weakly compact cardinals exist, $\mathrm{ZFC} \not\models \mathrm{TP}(\kappa) \to \mathtt{IC}(\kappa)$.
- Jensen (1972) showed that if $V = L$ then there is a κ-Aronszajn (in fact, Suslin) tree for every infinite successor cardinal κ.
  - This gives $\neg\mathrm{TP}(\kappa)$ for all successors κ, so $\mathrm{TP}(\kappa)$ implies κ a limit.
  - $V = L$ models GCH so all limits are strong limits, so $L \models \mathrm{TP}(\kappa) \to \mathtt{IC}(\kappa)$, so $\mathrm{ZFC} \not\models \mathrm{TP}(\kappa) \not\to \mathtt{IC}(\kappa)$.

$\mathtt{FP}(\kappa) \to \mathtt{IC}(\kappa)$?

I don't know whether $\mathtt{FP}(\kappa)$ implies inaccessibility.

What I know

- Drake (1974) gives $\mathtt{TP}(\kappa) \to \mathtt{FP}(\kappa)$ and $\mathtt{FP}(\kappa) \to \mathtt{WCT}(\kappa)$ under the assumption $\mathtt{IC}(\kappa)$, and this assumption is used in both proofs.
- Comfort, Negrepontis (1974) give κ weakly compact iff $\mathtt{FP}(\kappa)$ and $\kappa = \kappa^{<\kappa} = \bigcup_{\lambda<\kappa} \kappa^\lambda$.
  - $\kappa^{<\kappa}$ implies κ is weakly inaccessible.

Implications between characterizations

What implications do we know?

Assuming κ inaccessible, the following implications have "direct" proofs:

From WCT:	From PP:	From TP:	From EP:	From ID:
—	PP $\to$ WCT	TP $\to$ WCT	EP $\to$ WCT	
	—	TP $\to$ PP		
WCT $\to$ TP	PP $\to$ TP	—		ID $\to$ TP
WCT $\to$ EP		TP $\to$ EP	—	ID $\to$ EP
			EP $\to$ ID	—
		TP $\to$ FP		
	PP $\to$ OP			

From OP:	From FP:
OP $\to$ PP	FP $\to$ WCT

Next steps

Next steps

My next steps

- Writing my essay: Definitions and proofs.
- Attempt the other implications “directly”.

Priority questions

- Does FP imply inaccessibility?
- “Direct proofs” of related characterizations:
  - WCT2 $\to$ WCT
  - PP $\to$ PP2
- “Direct proofs” between WCT, PP, TP, EP:
  - WCT $\to$ PP
  - PP $\to$ EP
  - EP $\to$ PP
  - EP $\to$ TP

Thanks for listening!

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