Applications of Forcing: Baumgartner's Theorem

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Bases and downclasses

Given linear orders L_1, L_2, we write $L_1 \leq L_2$ if L_1 embeds into L_2.

Definition 1

Let $\mathcal{C}, \mathcal{B}, \mathcal{D}$ be classes of linear orders. We say that
$\mathcal{B}$ is a $\mathcal{C}$-basis if $\mathcal{B} \subseteq \mathcal{C}$ and $(\forall L \in \mathcal{C})(\exists L' \in \mathcal{B})(L' \leq L)$,and
$\mathcal{D}$ is a $\mathcal{C}$-downclass if $\mathcal{D} \subseteq \mathcal{C}$ and $(\forall L' \in \mathcal{D})(\forall L \in \mathcal{C})$ $(L \leq L' \Rightarrow L \in \mathcal{D})$.

Proposition 2

If $\mathcal{E}$ is a $\mathcal{C}$-downclass and $\mathcal{B}$ is a $\mathcal{C}$-basis, then $\mathcal{B} \cap \mathcal{E}$ is an $\mathcal{E}$-basis.
If $\mathcal{E}$ is a $\mathcal{C}$-basis and $\mathcal{B}$ is an $\mathcal{E}$-basis, then $\mathcal{B}$ is a $\mathcal{C}$-basis.

Definition 3

For any $\mathcal{E} \subseteq \mathcal{C}$, let $\mathcal{E}^\perp$ denote the downclass $\{L \in \mathcal{C} \mid (\forall L' \in \mathcal{E})(L' \nleq L)\}$.

Proposition 4

For any $\mathcal{E} \subseteq \mathcal{C}$, $\mathcal{E} \cup \mathcal{E}^\perp$ is a $\mathcal{C}$-basis.

Infinite linear orders

Any class containing finite linear orders admits a singleton basis.

Question 5

Does the class $\mathcal{C}_\infty$ of infinite linear orders admit a finite basis?

- The class of linear orders with order-type ω is a downclass in $\mathcal{C}_\infty$, so any basis must contain (an isomorphic copy of) ω.
- Similarly, any basis must also contain ω^*.

Proposition 6

$\{\omega, \omega^*\}$ *is a two-element basis for* $\mathcal{C}_\infty$*.*

Proof.

If $L \in \mathcal{C}_\infty$ satisfies $\omega^* \not\leq L$, then L is an infinite ordinal, and therefore contains an embedding of ω. □

Uncountable linear orders

Question 7

Does the class $\mathcal{C}_{unc}$ of uncountable linear orders admit a finite basis?

- Any basis for $\mathcal{C}_{unc}$ must contain (isomorphic copies of) ω_1, ω_1^*.
- However, $\{\omega_1, \omega_1^*\}$ is not a basis for $\mathcal{C}_{unc}$. Consider $\mathbb{R}$.

Definition 8

A linear order L is separable if it has a countable dense subset. Let $\mathcal{C}_{unc}^{sep}$ denote the class of uncountable separable linear orders.

- Clearly, ω_1 and ω_1^* are not separable, and separable linear orders are closed under suborders.
- So $\mathcal{C}_{unc}^{sep} \subseteq \{\omega_1, \omega_1^*\}^\perp$ is a $\mathcal{C}_{unc}$-downclass, and we must construct a $\mathcal{C}_{unc}^{sep}$-basis.

Questions

Question 9

Does the class $\mathcal{C}_{unc}^{sep}$ of uncountable separable linear orders admit a finite basis?

Question 10

Is $(\{\omega_1, \omega_1^*\} \cup \mathcal{C}_{unc}^{sep})^\perp \subseteq \mathcal{C}_{unc}$ empty? Does it admit a finite basis?

An additional assumption is necessary to obtain positive answers to these questions.

Theorem 11 (Sierpinski, 1950)

For any $X \in [\mathbb{R}]^{2^{\aleph_0}}$, there exists $Y \in [X]^{2^{\aleph_0}}$ such that $X \not\leq Y$.

Corollary 12

(CH) There is no finite basis for $\mathcal{C}_{unc}^{sep}$.

Proper forcing axiom

Definition 13

The proper forcing axiom (PFA) is the following strengthening of MA(ω_1): If $\mathbb{P}$ is a proper forcing and $\mathcal{D}$ is a family of $\aleph_1$-many dense subsets of $\mathbb{P}$, then there is a filter $G \subseteq \mathbb{P}$ that intersects each $D \in \mathcal{D}$.

For this presentation, we need only know that

- PFA is a reasonable assumption (it is equiconsistent with the existence of a supercompact cardinal),
- ccc and countably closed forcings are proper, and
- properness is preserved under single-step iteration.

PFA and the basis problem

Theorem 14 (Baumgartner)

(PFA) There exists a singleton basis for $\mathcal{C}_{unc}^{sep}$.

Theorem 15 (Moore, 2006)

(PFA) There exists a two-element basis for the class $(\{\omega_1, \omega_1^*\} \cup \mathcal{C}_{unc}^{sep})^\perp$.

Corollary 16

(PFA) There exists a five-element basis for $\mathcal{C}_{unc}$.

Uncountable separable linear orders

Definition 17

A linear order is $\aleph_1$-dense if there are exactly $\aleph_1$ points between any two distinct points.

Proposition 18

Any uncountable separable linear order contains an $\aleph_1$-dense suborder.

Theorem 19 (Baumgartner, 1973)

It is consistent that any two $\aleph_1$-dense separable linear orders are isomorphic.

Corollary 20

It is consistent that the class of uncountable separable linear orders has a singleton basis.

Baumgartner's lemma

Let $(Fn_<(A,B),\supseteq)$ be the poset of finite, monotone $f\colon (A,<)\to(B,<)$.

Lemma 21 (Baumgartner)

(CH) If A and B are $\aleph_1$-dense subsets of $\mathbb{R}$, then there is a ccc poset $\mathbb{P}(A,B)\subseteq Fn_<(A,B)$ such that $\Vdash_{\mathbb{P}(A,B)} A\cong B$.

We will recursively construct $\mathcal{A}=\langle A_\alpha \mid \alpha<\omega_1\rangle$, $\mathcal{B}=\langle B_\alpha \mid \alpha<\omega_1\rangle$ with

1. $\bigcup_{\alpha<\omega_1} A_\alpha = A$ and $\bigcup_{\alpha<\omega_1} B_\alpha = B$,

and obtain $\mathbb{P}(A,B)$ as the poset $Fn_<(\mathcal{A},\mathcal{B})$ of finite, monotone, partition-preserving functions.

To avoid obvious antichains and to ensure $\Vdash_{Fn_<(\mathcal{A},\mathcal{B})} A\cong B$:

2. Each A_α, B_α is countable and dense in $\mathbb{R}$.

To avoid the other antichains, we will **diagonalize**:

3. For each uncountable antichain $U\subseteq Fn_<(A,B)$, there is $\beta<\omega_1$, such that we avoid U while constructing A_α, B_α whenever $\alpha>\beta$.

Outline of the construction

- Requirements 1 and 2 are easy to satisfy.
- Too many antichains to diagonalize individually against!
- Instead, we will diagonalize against closures of antichains under a natural topology.
- Separability and CH imply that such closures can be enumerated $\langle c_\beta \mid \beta < \omega_1 \rangle$.
- At stage α, we will construct A_α, B_α while avoiding c_β for every $\beta < \alpha$.
- We may fail sometimes, but this does not create uncountable antichains, because of a minimality argument.

The construction

Fix a well-ordering $\prec$ on $\mathbb{R}$ with $(\mathbb{R}, \prec) \cong (\omega_1, \in)$ and an enumeration $\langle I_n \mid n < \omega \rangle$ of the rational intervals.

Stage α. Assume α is even (the odd case is symmetrical).

We will obtain A_α, B_α as $\{a_n \mid n < \omega\}, \{b_n \mid n < \omega\}$ respectively, after constructing $a_0, b_0, a_1, b_1, a_2, b_2, \dots$ inductively as follows.

- Let a_0 be the $\prec$-least element of $A \setminus \bigcup_{\beta<\alpha} A_\beta$.
- Having constructed $a_0, b_0, \dots, a_n$ for some $n < \omega$, let b_n be any element in

$$A \cap I_n \setminus [\bigcup_{q,a,\beta} X_{q,a,\beta} \cup \bigcup_{\beta<\alpha} B_\beta],$$

  where, for any q in **the already constructed part of** $Fn_<(\mathcal{A}, \mathcal{B})$, $a \in A$, $\beta < \alpha$,

$$X_{q,a,\beta} := \begin{cases} \{b \in B \mid q \cup \{(a,b)\} \in c_\beta\} & \text{if countable} \\ \varnothing & \text{otherwise} \end{cases}.$$

- Construction of a_{n+1} is symmetrical, using $X^{q,b,\beta}$'s and A_β's.

Checking ccc

Definition 22

Say that an uncountable antichain $U \subseteq Fn_<(\mathcal{A}, \mathcal{B})$ *is minimal if for some* $n \in \mathbb{N}$*, for every* $p, q \in U$

- $dom(p) \cap dom(q) = rng(p) \cap rng(q) = \varnothing$,
- $|p| = |q| = n$ *(i.e.,* $U \subseteq Fn_<(\mathcal{A}, \mathcal{B}) \cap [A \times B]^n$*), and*
- *there is no uncountable antichain in* $Fn_<(\mathcal{A}, \mathcal{B}) \cap [A \times B]^{<n}$ *(i.e., n is minimal).*

Proposition 23

Suppose $U \subseteq Fn_<(\mathcal{A}, \mathcal{B})$ *is a minimal uncountable antichain, with closure* c_β*. There exists* $q \cup \{(a, b)\} \in U$ *such that:*

- (a, b) *was constructed later than anything in q, and at a stage later than* β*, and*
- $X_{q,a,\beta}$ *and* $X^{q,b,\beta}$ *are countable.*

Baumgartner's theorem using iterated forcing

Corollary 24

If $M \models ZFC + CH +$ "A, B are $\aleph_1$-dense subsets of $\mathbb{R}$", G is $\mathbb{P}(A, B)$-generic over M, then $M[G] \models ZFC + CH +$ "A, B are $\aleph_1$-dense subsets of $\mathbb{R}$" $+ A \cong B$.

We wish to repeat this for all A, B.

There are $\aleph_2$ such pairs in M, so we will construct a $\subseteq$-increasing sequence $\langle M_\alpha \mid \alpha \leq \omega_2 \rangle$ such that for every $\alpha < \omega_2$,

1. $M_{\alpha+1} = M_\alpha[G]$ for some G that is $\mathbb{P}(A_\alpha, B_\alpha)$-generic over M_α for some $(A_\alpha, B_\alpha$ $\aleph_1$-dense in $\mathbb{R})^{M_\alpha}$, and
2. $M_\alpha \models ZFC + 2^{\aleph_0} = \aleph_1 + 2^{\aleph_1} = \aleph_2$,
3. for every $(A, B$ $\aleph_1$-dense in $\mathbb{R})^{M_{\omega_2}}$, there is $\alpha < \omega_2$ with $(A, B) = (A_\alpha, B_\alpha) \in M_\alpha$.

Outline of the iteration

We use a forcing iteration $\langle \mathbb{P}_\alpha \mid \alpha \leq \omega_2 \rangle$ where

1. $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathring{\mathbb{Q}}_\alpha$, $\Vdash_{\mathbb{P}_\alpha}$ "$\mathring{\mathbb{Q}}_\alpha$ is $\mathbb{P}(\mathring{A}_\alpha, \mathring{B}_\alpha)$ for some $\mathring{A}_\alpha, \mathring{B}_\alpha$ $\aleph_1$-dense in $\mathbb{R}$".
2. Limits stages use finite support.

Then each $\mathbb{P}_\alpha$ is ccc, 1 and 2 are satisfied, and $(|[\mathbb{R}]^{\aleph_1}| = \aleph_2)^{M_\alpha}$.
To satisfy 3, we choose A_α, B_α carefully.

3. Fix a bijection $b : \omega_2 \to \omega_2 \times \omega_2$ such that for every $\alpha < \omega_2$, there is $\beta \leq \alpha$ with $b(\alpha) = (\beta, \gamma)$.
   At stage α,
   - Fix a $\mathbb{P}_\alpha$-name $\mathring{f}_\alpha$ such that $\Vdash_{\mathbb{P}_\alpha}$ "$\mathring{f}_\alpha$ enumerates all pairs of $\aleph_1$-dense subsets of $\mathbb{R}$ in order type ω_2".
   - Let $b(\alpha) = (\beta, \gamma)$ and (A_α, B_α) be the pair $f_\beta(\gamma)$.

The following observation completes the proof.

Proposition 25

Every $\aleph_1$-dense subset of $\mathbb{R}$ in M_{ω_2} appears in M_α for some stage $\alpha < \omega_2$.

Baumgartner's theorem using PFA

Suppose $M \vDash ZFC + PFA$, and $A, B \in M$ are $\aleph_1$-dense suborders of $\mathbb{R}$.

Let $\mathbb{Q} := Coll(\omega_1, 2^{\aleph_0})$ and $(\mathbb{P}(A,B))^\circ$ be a $\mathbb{Q}$-name such that $\Vdash_{\mathbb{Q}}$ "$(\mathbb{P}(A,B))^\circ$ is as in Baumgartner's Lemma". Since $\mathbb{Q}$ is countably closed & $\Vdash_{\mathbb{Q}} (\mathbb{P}(A,B))^\circ$ is ccc, $\mathbb{P} := \mathbb{Q} * (\mathbb{P}(A,B))^\circ$ is proper.

Fix a $\mathbb{P}$-name $\mathring{f}$ such that $\Vdash_{\mathbb{P}} \mathring{f} : A \to B$ is an isomorphism. For each $a \in A$, $b \in B$, we have dense sets $D_a := \{p \in \mathbb{P} \mid p \Vdash \check{a} \in \mathrm{dom}(\mathring{f})\}$ and $D^b := \{p \in \mathbb{P} \mid p \Vdash \check{b} \in \mathrm{rng}(\mathring{f})\}$ in M.

By PFA, there is a filter $H \in M$ intersecting each D_a, D^b. Then $F := \{(a,b) \mid a \in A, b \in B, (\exists p \in H)(p \Vdash_{\mathbb{P}} \mathring{f}(\check{a}) = \check{b})\} \in M$ is an isomorphism $A \to B$.

Aronszajn lines

We now turn to constructing a finite basis for the remainder, $(\{\omega_1, \omega_1^*\} \cup \mathcal{C}_{unc}^{sep})^\perp$.

Definition 26

A linear order with cardinality $\aleph_1$ is called an Aronszajn line if it has no suborders isomorphic to ω_1, ω_1^* or any uncountable separable linear order. Let $\mathcal{C}_{unc}^{A}$ denote the class of Aronszajn lines.

$\mathcal{C}_{unc}^{A}$ is both a basis and a downclass of $(\{\omega_1, \omega_1^*\} \cup \mathcal{C}_{unc}^{sep})^\perp$, so it is both sufficient and necessary to construct basis for $\mathcal{C}_{unc}^{A}$.

Theorem 27 (Kurepa, 1936)

Aronszajn lines exist.

Countryman lines

Definition 28

A linear order L with cardinality $\aleph_1$ is called a Countryman line if $L \times L$ can be partitioned into countably many chains. Let $\mathcal{C}_{unc}^{C}$ be the class of Countryman lines.

- Suborders of Countryman lines are Countryman.
- Neither ω_1 nor ω_1^* are Countryman. No uncountable separable linear order is Countryman.
- Countryman lines are Aronszajn. Hence $\mathcal{C}_{unc}^{C}$ is a downclass of $\mathcal{C}_{unc}^{A}$.
- If L is Countryman, then no uncountable linear order embeds into both L and L^*.

Theorem 29 (Shelah, 1976)

Countryman lines exist.

Corollary 30

Aronszajn lines do not have a singleton basis.

Shelah's conjectures

Conjecture 31 (Shelah, 1976)

- *Any Aronszajn line contains a Countryman suborder.*
- *If L_1, L_2 are Countryman, then there is $L_1' \in \{L_1, L_1^*\}$ such that some uncountable linear order embeds into both L_1' and L_2.*

Theorem 32 (Moore, 2006)

(PFA) Whenever L_1 is Countryman and L_2 is Aronszajn, there is some $L_1' \in \{L_1, L_1^*\}$ that embeds into L_2.

Corollary 33

(PFA) The set $\{\omega_1, \omega_1^*, X, C, C^*\}$, where X is subset of $\mathbb{R}$ with cardinality $\aleph_1$, and C is any Countryman line, is a basis for the class of uncountable linear orders.

Thank You!