Co-analytic families of functions

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Joint work with Julia Millhouse

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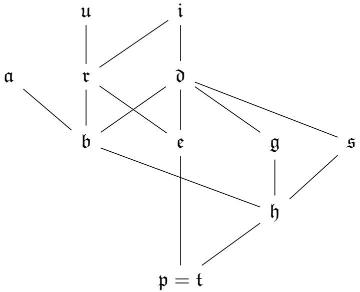
2. Usually special means maximal with respect to some combinatorial or topological property.
3. The minimal sizes of such special families are called cardinal characteristics and their relations give rise to a very rich and complicated theory:



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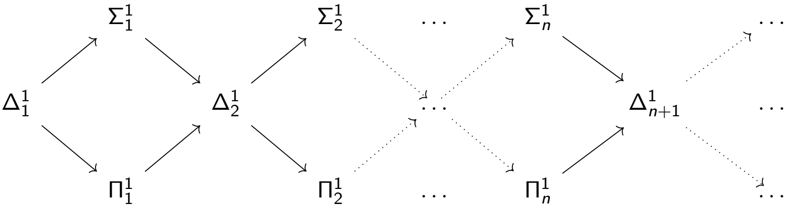

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1. What is the minimal complexity of various special families of reals?
2. In models separating cardinal characteristics $\mathfrak{x} < \mathfrak{y}$, can we additionally have witnesses for $\mathfrak{x}$ and $\mathfrak{y}$ of minimal complexity?
3. Given a special family of some complexity, can we construct a special family of lower complexity from it?

Example: Definability of mad families

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Theorem (Miller, 1989, [10])

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This result was improved in many papers to obtain various co-analytic forcing indestructible mad families. In particular we have

Theorem (Bergfalk, Fischer, Switzer, 2022, [2])

In L *there is a co-analytic tight mad family.*

Example: Definability of mad families - a shortcut

For example this implies the following for the Miller model over L:

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Consistently, $\aleph_1 = \mathfrak{a} < \mathfrak{d} = \aleph_2$ *and there is a* Π^1_1 *witness for* $\mathfrak{a} = \aleph_1$.

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However, Törnquist discovered the following shortcut:

Theorem (Törnquist, 2013 [13])

If there is a Σ^1_2 *mad family, then there is* Π^1_1 *mad family of the same size.*

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In particular, a Σ^1_2 witness for $\mathfrak{a}$ immediately implies a Π^1_1 witness for $\mathfrak{a}$. This gives an easier proof for the existence of a co-analytic mad family in L, but also for forcing extensions over L...

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Let $\mathbb{P}$ be a forcing, for which we know how to construct a $\mathbb{P}$-indestructible mad family (Sacks, Miller, Laver, random, etc.).

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Hence, in $\mathbb{P}$-extensions over L there is a co-analytic witness for $\mathfrak{a} = \aleph_1$.

Shortcuts for other families

Analogous shortcuts of the form $\Sigma^1_2 \implies \Pi^1_1$ are now known for others types of families, e.g.

1. maximal independent families - $\mathfrak{i}$
   (Brendle, Fischer, Khomskii, 2018, [7]),
2. maximal eventually different families of functions - $\mathfrak{a}_e$
   (Fischer, Schrittesser, 2021, [4]),
3. towers - $\mathfrak{t}$
   (Fischer, Schilhan, 2021, [3]),
4. Hausdorff gaps
   (Millhouse, 2024, [11])

Our goal was to prove similar shortcuts for various relatives of $\mathfrak{a}_e$ and the ideal-independence number $\mathfrak{s}_{mm}$.

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5. Basically, for every $A \in [\omega]^\omega$ and $c \in {}^\omega 2$ we need to be able to definably reconstruct A and c from $\Phi(A, c)$.
6. Finally, the Spector-Gandy-Theorem shows that $\mathcal{B}$ is in fact Π^1_1:

$$B \in \mathcal{B} \iff \exists (A, c) \in \Delta^1_1(B)\,((A, c) \in H \text{ and } \Phi(A, c) = B)$$

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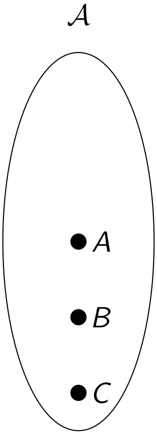

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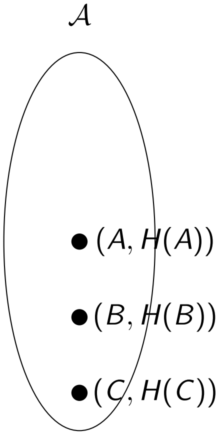

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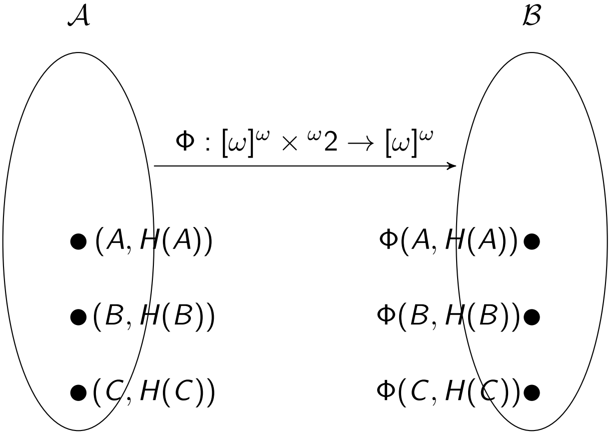

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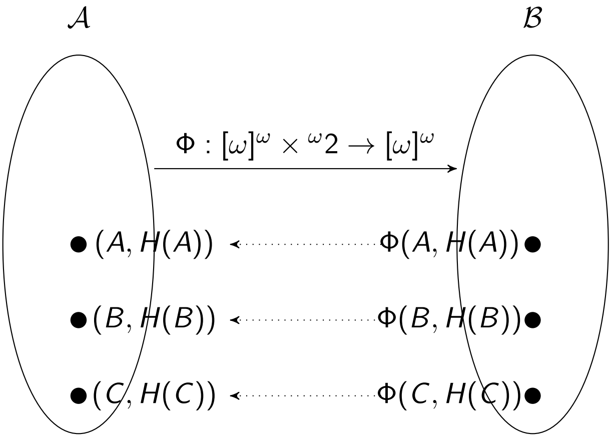

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Definition

$$\mathfrak{a}_\mathsf{e} := \min \{|F| \mid F \text{ is a maximal eventually different family}\}$$

Further, a maximal eventually different family of size $\mathfrak{a}_\mathsf{e}$ is said to be a witness for that cardinal characteristic.

Eventually different families of permutations

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Again, we define the cardinal invariant:

$$\mathfrak{a}_\mathrm{p} := \min\{|F| \mid F \text{ is a maximal eventually different family of permutations}\}$$

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The goal of this talk is to prove the following theorem:

Theorem (S., Millhouse, 2025)

If there is a Σ^1_2 maximal eventually different family of permutations, then there is a Π^1_1 maximal eventually different family of permutations of the same size.

Proof of the Theorem

1. Fix a Σ^1_2 maximal eventually different family F and a Π^1_1 partial function $H \subseteq {}^\omega\omega \times {}^\omega 2$ as before.

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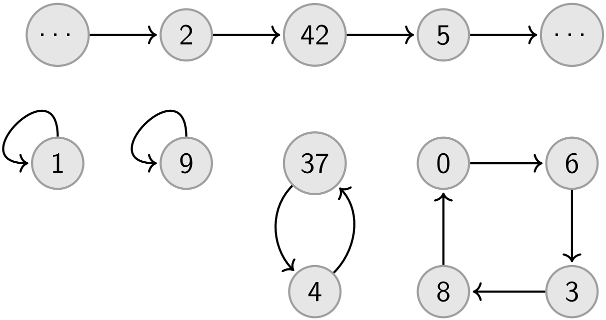
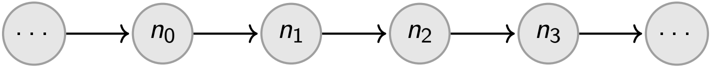
2. Remember, we have to construct a continuous map $\Phi : \omega^\omega \times {}^\omega 2 \to \omega^\omega$ which is recursively invertible and such that $G = \Phi[H]$ is a maximal eventually different family of permutations.
3. We may visualize a permutation f as the union of chains and cycles:



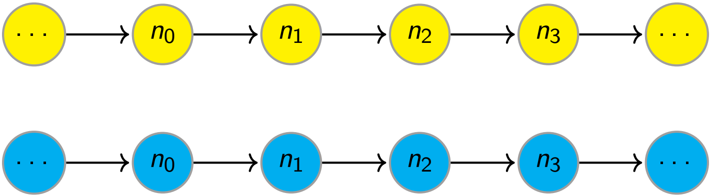
Visualization of the function Φ

For simplicity we just consider one chain



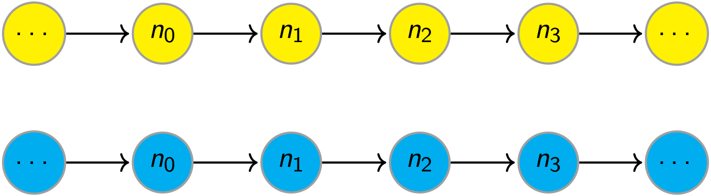
Visualization of the function Φ

First we duplicate both the domain and range of the function, but keep the structure of f on both copies. (Thus, the new domain and range is $\omega \times 2$)



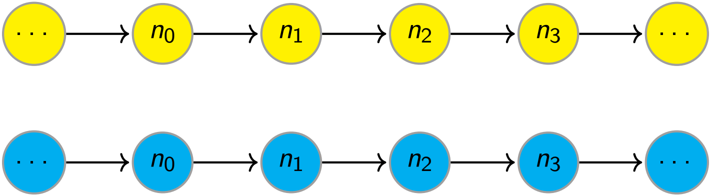
Visualization of the function Φ

Next, we need to adapt this permutation, so that it codes a real $c \in 2^\omega$, but still is a permutation and keeps the structure of f:



Visualization of the function Φ

Key Idea: At place n flip both functions values iff $c(n) = 1$:



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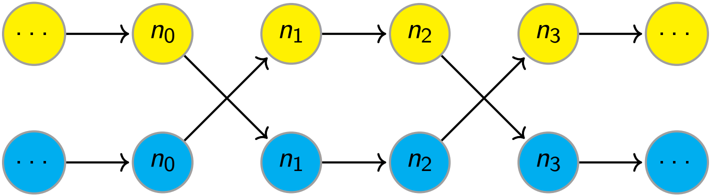
For example, if $c(n_0) = 1$, $c(n_1) = 0$ and $c(n_2) = 1$ we get:



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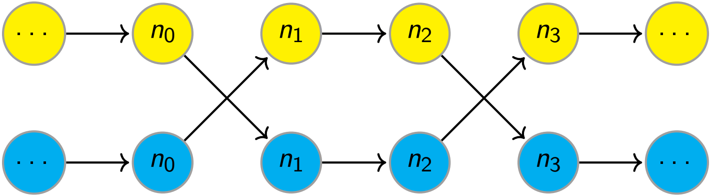
This gives us $\Phi(f, c)$ from which we can clearly decode f and c again. It is also easy to see that $\Phi[H]$ will still be eventually different.



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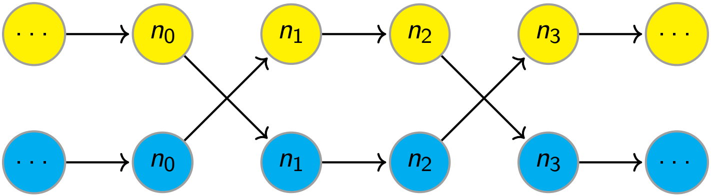
The more difficult part of the proof is the maximality of $\Phi[H]$. In fact, $\Phi[H]$ will NOT be maximal!



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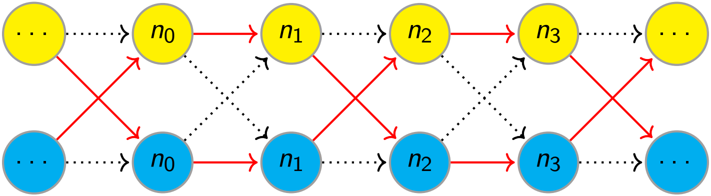
Note that $\Phi(f, 1-c)$ is everywhere different from $\Phi(f, c)$:



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We will define a certain permutation $\tilde{h}$ on ω from h.

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$$\begin{array}{ccl} \mathbf{G} \text{ (on } \omega \times 2) & & h \\ & & \downarrow \\ \mathbf{F} \text{ (on } \omega) & & \tilde{h} \longrightarrow f \end{array}$$

By maximality of F choose $f \in F$ with $\tilde{h} =^\infty f$.

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& \downarrow & & \uparrow \\
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\end{array}$$

Finally, we will either have $h =^\infty \Phi(f, H(f))$ or $h =^\infty \Phi^*(f, H(f))$.

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4. Instead we will find a function $i : \omega \to 2$, such that $\tilde{h}(n) := p_0(h(n, i(n)))$ defines a permutation.

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4. Instead we will find a function $i : \omega \to 2$, such that $\tilde{h}(n) := p_0(h(n, i(n)))$ defines a permutation.
5. To this end, note that the function $g : \omega \times 2 \to \omega$ defined by $g(n, i) := p_0(h(n, i))$ is 2-to-1, i.e. every $n \in \omega$ has exactly two preimages.

Crucial lemma

Lemma

Assume that $g : \omega \times 2 \to \omega$ *is 2-to-1. Then there is a function* $i : \omega \to 2$*, such that* $\tilde{h}(n) := g(n, i(n))$ *is a permutation.*

Proof.

We define a bipartite graph with possible multi-edges:

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1. We have countably many left $\{L_n \mid n \in \omega\}$ and right $\{R_n \mid n \in \omega\}$ nodes.

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We define a bipartite graph with possible multi-edges:

1. We have countably many left $\{L_n \mid n \in \omega\}$ and right $\{R_n \mid n \in \omega\}$ nodes.
2. For each $n \in \omega$ and $i \in 2$ we have an edge $e_{n,i}$ between L_n and $R_{g(n,i)}$.

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2. For each $n \in \omega$ and $i \in 2$ we have an edge $e_{n,i}$ between L_n and $R_{g(n,i)}$.

By construction, every L_n has degree 2. Since g is 2-to-1, also every R_n has degree 2, i.e. the graph is 2-regular.

Crucial lemma

Reminder: For $n \in \omega$ and $i \in 2$ we have an edge $e_{n,i}$ between L_n and $R_{g(n,i)}$.

Proof.

Choose a perfect matching P of our graph and define

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Finally, it remains to see that $\tilde{h}(n) := g(n, i(n))$ is a permutation.

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Reminder: For $n \in \omega$ and $i \in 2$ we have an edge $e_{n,i}$ between L_n and $R_{g(n,i)}$, $i(n) := i$ iff $e_{n,i} \in P$ and $\tilde{h}(n) = g(n, i(n))$.

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So assume $\tilde{h}(n) = g(n, i(n)) = g(m, i(m)) = \tilde{h}(m)$.

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Now, let $m \in \omega$. Since, P is perfect there is an edge $e_{n,i}$ incident to R_m in P. But $e_{n,i}$ is incident $R_{g(n,i)}$, so we have $g(n, i(n)) = m$. Hence, $\tilde{h}(n) = m$ and $\tilde{h}$ is surjective. □

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5. Let $n \in A$, then there is a $j \in 2$ with

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6. Hence, either

$$h(n, i(n)) = \Phi(f, c)(n, i(n)) \quad \text{or} \quad h(n, i(n)) = \Phi^*(f, c)(n, i(n)).$$

Reduction theorem for cofinitary groups

This finishes the proof of our theorem:

Theorem (S., Millhouse, 2025)

If there is a Σ^1_2 maximal eventually different family of permutations, then there is a Π^1_1 maximal eventually different family of permutations of the same size.

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We also came up with other ideas to obtain similar results for

1. Van Douwen families,
2. free generating sets of (strongly) maximal cofinitary groups,
3. maximal ideal independent families.

Van Douwen families

Definition

An eventually different family of functions $F \subseteq {}^{\omega}\omega$ is called **Van Douwen** iff for every infinite partial function $h : \omega \xrightarrow{\text{partial}} \omega$ there is an $f \in F$ with $h =^{\infty} f$.

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Cofinitary groups

Let $G \subseteq S_\omega$ be an eventually different family of permutations. Then, G need not be a group. It may fail to be closed under inverses or concatenation.

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$$\mathfrak{a}_\mathrm{g} := \min \{|G| \mid G \text{ is a maximal cofinitary group}\}$$

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Observation

A group $G \subseteq S_\omega$ is cofinitary iff every element is either the identity or only has finitely many fixpoints.

Definability of cofinitary groups

Definition

Let $G \subseteq S_\omega$ be a group and $G_0 \subseteq G$. Then G is a **generating set** if $G = \langle G_0 \rangle$, where $\langle G_0 \rangle$, is the group generated by G_0.

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For cofinitary groups its often easier to provide a definable generating set instead of the whole group. If G_0 is definable, also G is definable from G_0, however its complexity may be higher.

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For cofinitary groups its often easier to provide a definable generating set instead of the whole group. If G_0 is definable, also G is definable from G_0, however its complexity may be higher.

Observation

Recursively, it is easy to construct a Σ^1_2 generating set for a maximal cofinitary group in L using the Δ^1_2 well-order.

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Theorem (Horowitz, Shelah, 2016 [6])

In ZF *there is a Borel maximal cofinitary group (and a Borel maximal eventually different family).*

Reduction theorem for cofinitary groups

Theorem (S., Millhouse, 2025)

If there is a Σ^1_2 generating set for a free maximal cofinitary group, which is also maximal as an eventually different family of permutations, then there also is a Π^1_1 generating set for a maximal cofinitary group of the same size, which is also maximal as an eventually different family of permutations.

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Corollary

If there is a Σ^1_2 generating set for a free tight cofinitary group, then there is a Π^1_1 generating set for a maximal cofinitary group of the same size.

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Question

If there is a Σ^1_2 maximal cofinitary group, is there is a Π^1_1 maximal cofinitary group of the same size?

Proof sketch

1. Fix a Σ^1_2 generating set F for a free maximal cofinitary group $G = \langle F \rangle$, which is also maximal as an eventually different family of permutations and a Π^1_1 partial function $H \subseteq {}^\omega\omega \times {}^\omega 2$ as before.

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2. Again, we define $\hat{F} := \Phi[H] \cup \Phi^*[H]$ and let $\hat{G} = \langle \hat{F} \rangle$.

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\end{array}$$

Crucially, we can find a natural surjective group homomorphism $\Psi : \hat{G} \to G$ and compute its kernel as $\{\mathrm{id}, \tau\}$, where $\tau(n, i) := (n, 1 - i)$ is the flip map. This also implies that $\hat{G} \cong G \times \mathbb{Z}/2$.

Ideal independent families

Definition

A family $\mathcal{I} \subseteq [\omega]^\omega$ is called ideal independent iff for every $A \in \mathcal{I}$ and $\mathcal{I}_0 \in [\mathcal{I}]^{<\omega}$ we have that $A \not\subseteq^* \bigcup \mathcal{I}_0$.

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Definition

Again, we define a cardinal invariant:

$$\mathfrak{s}_{\mathrm{mm}} := \min \{|\mathcal{I}| \mid \mathcal{I} \text{ is a maximal ideal independent family}\}$$

Observation

Mad families and maximal independent families are ideal independent, but neither have to be maximal as an ideal independent family.

Reduction theorem for ideal independent families

Theorem (Bardyla, Cancino, Fischer, Switzer, 2025, [1])

(CH) There is a maximal ideal independent family indestructible by any proper, ${}^{\omega}\omega$-bounding, p-point preserving forcing.

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Corollary

Consistently, $\aleph_1 = \mathfrak{s}_{mm} < \mathfrak{c}$ with a Π^1_1 witness for $\mathfrak{s}_{mm}$.

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2. Define a function $\Psi : [\omega]^\omega \times [\omega]^\omega \to [\omega]^\omega$ and $z \in [\omega]^\omega$ by

$$\begin{aligned} \Psi(x, c) &:= 3x \cup (3c + 1) \\ z &:= \qquad (3\omega + 1) \cup (3\omega + 2) \end{aligned}$$

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1. Fix a Σ^1_2 maximal ideal independent family $\mathcal{I}$ and a Π^1_1 partial function $H \subseteq [\omega]^\omega \times [\omega]^\omega$ as before.
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$$\begin{aligned} \Psi(x, c) &:= 3x \cup (3c + 1) \\ z &:= \qquad (3\omega + 1) \cup (3\omega + 2) \end{aligned}$$

1. Define $\mathcal{J} := \Psi[H] \cup \{z\}$.

Proof sketch

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1. Define $\mathcal{J} := \Psi[H] \cup \{z\}$.
2. It is not too difficult to show that $\mathcal{J}$ is maximal ideal independent and Π^1_1.

Summary

Invariant	**Borel**	Π^1_1 + $\neg$**CH**	$\Sigma^1_2 \Rightarrow \Pi^1_1$
a	No	Yes	Yes
a_v	No	Yes	?
a_e	Yes	Yes	Yes
a_p	Yes	Yes	?
a_g	Yes	Yes	?
s_{mm}	?	?	?

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a_p	Yes	Yes	Yes
a_g	Yes	Yes	Yes*
s_{mm}	?	Yes	Yes

*under extra assumptions and only for the generators

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Thank you for your attention!