Independence in Higher Baire Spaces

Anna Lenz

University of Hamburg

~~Vienna, Faculty of Mathematics~~

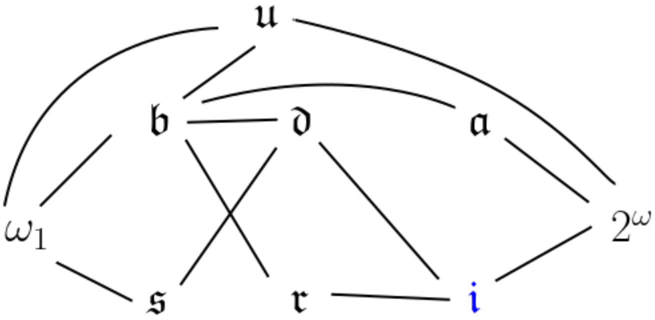
Supervisor: Assoz.-Professor Vera Fischer, Privatdoz., PhD.

November, 27th

~~June, 18th~~ 2025

Why are we interested?

- Understand models in which *CH* or *GCH* does not hold
- Many open questions



Goals

- *con*($\mathfrak{i} < \mathfrak{c}$).
- *con*($\mathfrak{i}(\kappa) < 2^\kappa$)

The countable independence number

Definition

A family $\mathcal{A} \subseteq [\omega]^\omega$ is called *independent* if for any two finite disjoint subfamilies $B, C \subseteq \mathcal{A}$ it holds that $|\bigcap_{A \in B} A \setminus (\bigcup_{A \in C} A)| = \omega$.

Example

$\mathcal{A} = \{A_p : p \in \omega \text{ is a prime number}\}$ with
$A_p = \{n \in \omega : \exists k \text{ s.t } n = k \cdot p\}$ is a countable independent family.

$A_3 = \{3, 6, 9, 12, \ldots\}$

The countable independence number

Definition

A family $\mathcal{A} \subseteq [\omega]^\omega$ is called *independent* if for any two finite disjoint subfamilies $B, C \subseteq \mathcal{A}$ it holds that $|\bigcap_{A \in B} A \setminus (\bigcup_{A \in C} A)| = \omega$.

Definition

- An independent family is called *maximal* if it is maximal under set inclusion.
- $\mathfrak{i} := \min\{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^\omega \text{ is maximal independent}\}$

Notation

If $A \subseteq \omega$, let $A^0 := A$ and $A^1 := \omega \setminus A$.

For $\mathcal{A} \subseteq [\omega]^\omega$ let $FF(\mathcal{A})$ consist of all partial functions h, such that

- $dom(h) \subseteq \mathcal{A}$, finite
- $im(h) \subseteq \{0, 1\}$

Then write $\mathcal{A}^h$ for $\bigcap\{A^{h(A)} \mid A \in dom(h)\}$.

Notation

If $A \subseteq \omega$, let $A^0 := A$ and $A^1 := \omega \setminus A$.

For $\mathcal{A} \subseteq [\omega]^\omega$ let $FF(\mathcal{A})$ consist of all partial functions h, such that

- $dom(h) \subseteq \mathcal{A}$, finite
- $im(h) \subseteq \{0, 1\}$

Then write $\mathcal{A}^h$ for $\bigcap\{A^{h(A)} \mid A \in dom(h)\}$.

Reformulation

A family $\mathcal{A}$ is independent if for all $h \in FF(\mathcal{A})$, $|\mathcal{A}^h| = \omega$.

$\mathfrak{i}$ is well defined!

Theorem

Every independent family is contained in a maximal independent family.

Proof

- Apply Zorn's Lemma
- Let $(\mathcal{A}_i)_{i<\alpha}$ increasing chain of independent families.
- $\mathcal{A} := \bigcup_{i<\alpha} \mathcal{A}_i$
- Let $h \in FF(\mathcal{A})$ then $dom(h) \subseteq \mathcal{A}_i$ for some $i < \alpha$
- Hence, $\mathcal{A}^h = \mathcal{A}_i^{h \restriction \mathcal{A}_i}$, which is infinite. $\square$

"Boolean Combination"

Some inequalities

Definition

- $\mathfrak{d}$ is the minimum cardinality of a dominating family
- $\mathfrak{r}$ is the minimum cardinality of an unsplit family

- $\omega < \mathfrak{i}$
- $\mathfrak{d} \leq \mathfrak{i}$
- $\mathfrak{r} \leq \mathfrak{i}$
- $\mathfrak{i} \leq 2^\omega$ - in fact there is a maximal independent family of size $\mathfrak{c}$

Geschke

Motivation

$con(\mathfrak{i} < \mathfrak{c})$

$$M \xrightarrow{\text{forcing}} M[G]$$

$\mathcal{I}$ max ind + something

$|\mathcal{I}| = \mathfrak{i}$

$\mathcal{I}$ max ind + something*

$|\mathcal{I}| < \mathfrak{c}$

Shelah 1992 $con(\mathfrak{i} < \mathfrak{c})$

Dense maximality and selective independent families

Definition

An independent family $\mathcal{A}$ is *densely maximal* if for every $X \subseteq \omega$ and every $h \in FF(\mathcal{A})$ there is $h' \supseteq h$, such that $|\mathcal{A}^{h'} \cap X| < \omega$ or $|\mathcal{A}^{h'} \setminus X| < \omega$.

$X \subseteq \omega \quad \exists h \ (\mathcal{A}^h \cap X = \emptyset \quad \mathcal{A}^h \cap (\omega \setminus X) = \emptyset)$

Dense maximality and selective independent families

Definition

An independent family $\mathcal{A}$ is *densely maximal* if for every $X \subseteq \omega$ and every $h \in FF(\mathcal{A})$ there is $h' \supseteq h$, such that $|\mathcal{A}^{h'} \cap X| < \omega$ or $|\mathcal{A}^{h'} \setminus X| < \omega$.

Definition

Let $\mathcal{A}$ be an independent family.

1. The *density filter* is the set modulo finite
   $fil(\mathcal{A}) := \{X \subseteq \omega \mid \mathcal{A}^h \subseteq^* X \text{ for all } h \in FF(\mathcal{A})\}$.
2. The *density ideal* is the set $\{X \subseteq \omega \mid \omega \setminus X \in fil(\mathcal{A})\}$.

Dense maximality and selective independent families

Lemma (STST 2024)

Let $\mathcal{A}$ be a (maximal) independent family. The following are equivalent:

1. $\mathcal{A}$ is densely maximal.
2. $\mathcal{P}(\omega) = \mathit{fil}(\mathcal{A}) \cup \langle\{\omega \setminus \mathcal{A}^h \mid h \in FF(\mathcal{A})\}\rangle_{dn}$. $\subseteq^*$
3. For every $h \in FF(\mathcal{A})$ and every $X \subseteq \mathcal{A}^h$ either $\mathcal{A}^h \setminus X$ is in the density ideal or there is $h' \supseteq h$ such that $\mathcal{A}^{h'} \subseteq \mathcal{A}^h \setminus X$.

Definition

Let $\mathcal{F} \subseteq \mathcal{P}(\omega)$ be a filter, then

- $\mathcal{F}$ is a *p-set* if for every countable $\mathcal{H} \subseteq \mathcal{F}$ there is $F \in \mathcal{F}$ such that $F \subseteq^* H$ for all $H \in \mathcal{H}$. pseudo-int
- $\mathcal{F}$ is a *q-set* if for every partition $\mathcal{E}$ of ω into finite sets there is $F \in \mathcal{F}$ such that $|F \cap E| \leq 1$ for every $E \in \mathcal{E}$
- $\mathcal{F}$ is *Ramsey* if it is a p-set and a q-set.

Dense maximality and selective independent families

Definition

Let $\mathcal{F} \subseteq \mathcal{P}(\omega)$ be a filter, then

- $\mathcal{F}$ is a *p-set* if for every countable $\mathcal{H} \subseteq \mathcal{F}$ there is $F \in \mathcal{F}$ such that $F \subseteq^* H$ for all $H \in \mathcal{H}$.
- $\mathcal{F}$ is a *q-set* if for every partition $\mathcal{E}$ of ω into finite sets there is $F \in \mathcal{F}$ such that $|F \cap E| \leq 1$ for every $E \in \mathcal{E}$
- $\mathcal{F}$ is *Ramsey* if it is a p-set and a q-set.

Definition

An independent family $\mathcal{A}$ is called *selective* if it is densely maximal and *fil*$(\mathcal{A})$ is Ramsey.

Dense maximality and selective independent families

Theorem (Shelah, 1992)

Selective independent families exist under CH.

Questions

- Does ZFC prove the existence of selective independent families?
- What about densely maximal ones?
- Is there a densely maximal independent family which is not selective?

Construction of a selective independent family

Definition

- Let $M \models CH$ c.t.m
- $\mathbb{P} := \{(\mathcal{A}, A) \mid \mathcal{A}$ countable independent family, $A \subseteq \omega$ and $|\mathcal{A}^h \cap A| = \omega$ for all $h \in FF(\mathcal{A})\}$.
- Let G be $\mathbb{P}$-generic and $\mathcal{A}_G := \bigcup\{\mathcal{A} \mid \exists A \subseteq \omega \text{ s.t } (\mathcal{A}, A) \in G\}$.

$(\mathcal{A}, A) \leq_{\mathbb{P}} (\mathcal{B}, B)$ $\quad \mathcal{A} \supseteq \mathcal{B}$

$A \leq^* B$

Construction of a selective independent family

Definition

- Let $M \models CH$ c.t.m $\leftarrow$ ω_1-closed ω_2-cc
- $\mathbb{P} := \{(\mathcal{A}, A) \mid \mathcal{A}$ countable independent family, $A \subseteq \omega$ and $|\mathcal{A}^h \cap A| = \omega$ for all $h \in FF(\mathcal{A})\}$.
- Let G be $\mathbb{P}$-generic and $\mathcal{A}_G := \bigcup\{\mathcal{A} \mid \exists A \subseteq \omega \text{ s.t } (\mathcal{A}, A) \in G\}$.

To show

1. $\mathcal{A}_G$ is an independent family ✓
2. $\mathcal{A}_G$ is densely maximal $\leadsto$ density + forcing ✓
3. $fil(\mathcal{A}_G)$ is a p-set
4. $fil(\mathcal{A}_G)$ is a q-set

$fil(\mathcal{A}_G)$ is a p-set!

Let $\mathcal{F}_G := \{A \mid \exists \mathcal{A}$ such that $(\mathcal{A}, A) \in G\}$

Lemma

$fil(\mathcal{A}_G)$ is generated by $\mathcal{F}_G \cup Fr$, in fact for every $A \in fil(\mathcal{A}_G)$ exists $F \in \mathcal{F}_G$ such that $F \subseteq^* A$. w/ finite sets

$\mathcal{H} \subseteq fil(\mathcal{A}_G)$ countable

for each $A \in \mathcal{H}$, $F_A \in \mathcal{F}_G$ $\quad F_A \subseteq^* A$

$fil(\mathcal{A}_G)$ is a p-set!

Let $\mathcal{F}_G := \{A \mid \exists \mathcal{A} \text{ such that } (\mathcal{A}, A) \in G\}$

Lemma

$fil(\mathcal{A}_G)$ is generated by $\mathcal{F}_G \cup Fr$, in fact for every $A \in fil(\mathcal{A}_G)$ exists $F \in \mathcal{F}_G$ such that $F \subseteq^* A$.

- Suffices to show any countable subset of $\mathcal{F}_G$ has a pseudo-intersection (in $M[G]$!).
- Assume not and let $\mathcal{H} \subset \mathcal{F}_G$ be countable subset without pseudo-intersection.
- Then there is $p \in G$ s.t $p \Vdash \forall X \in \mathcal{F}_G(\exists H \in \mathcal{H}(X \not\subseteq^* H))$.
- Enumerate $\mathcal{H} = \{H_i \mid i < \omega\}$, for every $i < \omega$ there is $\mathcal{A}_i$ s.t $(\mathcal{A}_i, H_i) \in G$. $q_0 \leq (\mathcal{A}_0, H_0), p$ $q_1 \leq (\mathcal{A}_1, H_1), q_0$
- Use that G is a filter and $\mathbb{P}$ is $< \omega_1$-closed to obtain $q \in \mathbb{P}$ s.t $q \leq (\mathcal{A}_i, H_i)$ for all $i < \omega$ and $q < p$. $\leq$ $\square$

$(\mathcal{B}, H)$

$\mathit{fil}(\mathcal{A}_G)$ is a q-set!

- Let $\mathcal{B} = (B_n)_{n<\omega}$ be a partition of ω into finite sets.
- We want to find a semiselector in $\mathcal{F}_G$.
- $\mathcal{D}_{\mathcal{B}} := \{(\mathcal{A}, A) \in \mathbb{P} \mid A \text{ is a semiselector for } \mathcal{B}\}$
- Now show that $\mathcal{D}_{\mathcal{B}}$ is dense!

$(\mathcal{A}, A) \in \mathbb{P}$ want to have $(\mathcal{C}, C) \leq (\mathcal{A}, A)$ s.t. $(\mathcal{C}, C) \in \mathcal{D}_{\mathcal{B}}$

$FF(\mathcal{A}) = \{h_n : n<\omega\}$ s.t. each $h \in FF(\mathcal{A})$ occurs infinitely often

$|\mathcal{A}^{h_0} \cap A| = \omega$ $\exists m_0 < \omega$ s.t. $\underbrace{\mathcal{A}^{h_0} \cap A \cap B_{m_0}}_{\ni a_0} \neq \emptyset$

$|\mathcal{A}^{h_1} \cap A| = \omega$ $\exists m_1 > m_0$ $\underbrace{\mathcal{A}^{h_1} \cap A \cap B_{m_1}}_{a_1} \neq \emptyset$

$|\mathcal{A}^{h_{n+1}} \cap A|$ $\exists m_{n+1} > m_n$ $\underbrace{\mathcal{A}^{h_{n+1}} \cap A \cap B_{m_{n+1}}}_{\ni a_{n+1}} \neq \emptyset$

$C = \{a_n : n<\omega\}$ $(\mathcal{A}, C)$

An equivalent condition to the q-set property

Theorem (Cruz-Chapital, Fischer, Guzmán, Šupina)

A filter $\mathcal{F}$ is a q-set if and only if for every strictly increasing function $f : \omega \to \omega$ there is $A \in \mathcal{F}$, such that if $\{a_n : n \in \omega\}$ is its enumeration, then $f(a_n) < a_{n+1}$ for all $n \in \omega$.

An equivalent condition to the q-set property

Theorem (Cruz-Chapital, Fischer, Guzmán, Šupina)

A filter $\mathcal{F}$ is a q-set if and only if for every strictly increasing function $f : \omega \to \omega$ there is $A \in \mathcal{F}$, such that if $\{a_n : n \in \omega\}$ is its enumeration, then $f(a_n) < a_{n+1}$ for all $n \in \omega$.

Caution

The original definition of a q-set does not generalise to higher Baire space!!!

$$f(n) \leq (f(a_n) < a_{n+1}$$

$$e_A(n+1)$$

$$f(n) \leq \cancel{a}\, e_A(n)$$

Sketch of proof of $con(\mathfrak{i} < 2^\omega)$

1. Start in a ground model V in which CH holds
2. Use the forcing poset
$\mathbb{P} := \{(\mathcal{A}, A) : \mathcal{A}$ is a countable independent family, $A \subseteq \omega$ and $|\mathcal{A}^h \cap A| = \omega$ for every $h \in FF(\mathcal{A})\}$ with
$(\mathcal{A}, A) \leq (\mathcal{B}, B)$ iff $\mathcal{A} \supseteq \mathcal{B}$ and $A \subseteq^* B$.
3. Let G be $\mathbb{P}$-generic and
$\mathcal{A}_G := \bigcup\{\mathcal{A} \mid \exists A \subseteq \omega \text{ s.t } (\mathcal{A}, A) \in G\}$.
4. Show that $\mathcal{A}_G$ is selective in $V[G]$.
5. Notice that $(2^\omega = \omega_1)^{V[G]}$
6. Show that $\mathcal{A}_G$ remains densely maximal after ω_2 many iterations of the Sacks forcing $\mathbb{S}$.

Independence in the higher Baire space

How can we define independent families of subsets of κ?

Definition (Strong independence)

- A family $\mathcal{A} \subseteq [\kappa]^\kappa$ is called strongly κ-independent if for every two disjoint $B, C \subseteq \mathcal{A}$ of size $< \kappa$, $|\bigcap_{A \in B} A \setminus (\bigcup_{A \in C} A)| = \kappa$
- A strongly κ-independent family is called maximal if it is maximal with respect to set inclusion.
- $\mathfrak{i}_s(\kappa) = \min\{\mathcal{A} \mid \mathcal{A}$ is maximal strongly κ-independent$\}$

How can we define independent families of subsets of κ?

Definition (Strong independence)

- A family $\mathcal{A} \subseteq [\kappa]^\kappa$ is called strongly κ-independent if for every two disjoint $B, C \subseteq \mathcal{A}$ of size $< \kappa$, $|\bigcap_{A \in B} A \setminus (\bigcup_{A \in C} A)| = \kappa$
- A strongly κ-independent family is called maximal if it is maximal with respect to set inclusion.
- $\mathfrak{i}_s(\kappa) = \min\{\mathcal{A} \mid \mathcal{A}$ is maximal strongly κ-independent$\}$

Caution

$\mathfrak{i}_s(\kappa)$ is not necessarily well-defined!!

$i_{\lambda,\kappa}$

$\mathfrak{i}$ is well defined!

Theorem

Every independent family is contained in a maximal independent family.

Proof

1. Apply Zorn's Lemma
2. Let $(\mathcal{A}_i)_{i<\alpha}$ increasing chain of independent families.
3. $\mathcal{A} := \bigcup_{i<\alpha} \mathcal{A}_i$
4. Let $h \in FF(\mathcal{A})$ then $dom(h) \subseteq \mathcal{A}_i$ for some $i < \alpha$
5. Hence, $\mathcal{A}^h = \mathcal{A}_i^{h \restriction \mathcal{A}_i}$, which is infinite. $\square$

Definition

- A family $\mathcal{A} \subseteq [\kappa]^\kappa$ is called κ-independent if for every two **finite**, disjoint $B, C \subseteq \mathcal{A}$, $|\bigcap_{A \in B} A \setminus (\bigcup_{A \in C} A)| = \kappa$
- A κ-independent family is called maximal if it is maximal with respect to set inclusion.
- $\mathfrak{i}(\kappa) = \min\{\mathcal{A} \mid \mathcal{A}$ is maximal κ-independent$\}$

κ regular uncountable

Definition

- A family $\mathcal{A} \subseteq [\kappa]^\kappa$ is called κ-independent if for every two **finite**, disjoint $B, C \subseteq \mathcal{A}$, $|\bigcap_{A \in B} A \setminus (\bigcup_{A \in C} A)| = \kappa$
- A κ-independent family is called maximal if it is maximal with respect to set inclusion.
- $\mathfrak{i}(\kappa) = \min\{\mathcal{A} \mid \mathcal{A}$ is maximal κ-independent$\}$

Notation

Write $FF(\mathcal{A})$ for all finite partial functions $\mathcal{A} \to \{0, 1\}$. With $A^0 =: A$ and $A^1 =: \kappa \setminus A$ define $\mathcal{A}^h$ as before.

Facts

- Every κ-independent family is contained in a maximal one
- There is a maximal independent family of size 2^κ
- $\kappa^+ \leq \mathfrak{i}(\kappa)$
- $\mathfrak{r}(\kappa) \leq \mathfrak{i}(\kappa)$

Questions

- Is $\mathfrak{d}(\kappa) \leq \mathfrak{i}(\kappa)$?
- Is there a relation between $\mathfrak{i}(\kappa)$ and $\mathfrak{i}_s(\kappa)$?

If $\mathcal{I}$ is strongly independent family
$|\mathcal{I}| < \mathfrak{d}(\kappa) \Rightarrow \mathcal{I}$ not maximal

- $\alpha^{<\kappa} < \mathfrak{d}(\kappa)$ for $\alpha < \mathfrak{d}(\kappa)$
- κ regular $(\kappa^{<\kappa} = \kappa)$

Facts

- Every κ-independent family is contained in a maximal one
- There is a maximal independent family of size 2^κ
- $\kappa^+ \leq \mathfrak{i}(\kappa)$
- $\mathfrak{r}(\kappa) \leq \mathfrak{i}(\kappa)$

Questions

- Is $\mathfrak{d}(\kappa) \leq \mathfrak{i}(\kappa)$?
- Is there a relation between $\mathfrak{i}(\kappa)$ and $\mathfrak{i}_s(\kappa)$?

Definition

An independent family $\mathcal{A}$ is *densely maximal* if for every $X \subseteq \kappa$ and every $h \in FF(\mathcal{A})$ there is $h' \supseteq h$, such that $\mathcal{A}^{h'} \cap X = \emptyset$ or $\mathcal{A}^{h'} \setminus X = \emptyset$.

Outline

Goal: prove *con*$(\mathfrak{i}(\kappa) < 2^\kappa)$

1. Assume that κ is measurable in our ground model V_0 and $2^\kappa = \kappa^+$ and let $\mathcal{U}$ be a (non-principal) normal measure on κ.

Outline

Goal: prove $con(\mathfrak{i}(\kappa) < 2^\kappa)$

1. Assume that κ is measurable in our ground model V_0 and $2^\kappa = \kappa^+$ and let $\mathcal{U}$ be a (non-principal) normal measure on κ.
2. Define a poset $\mathbb{P}_\mathcal{U}$, let G be $\mathbb{P}_\mathcal{U}$-generic and construct $\mathcal{A} = \mathcal{A}_G$ in the generic extension $V := V_0[G]$.
3. Show that $\mathbb{P}_\mathcal{U}$ preserves cardinals, $\mathcal{A}$ is densely maximal, and "*similar to selective*".
4. Find an equivalent condition to being densely maximal.
5. Introduce the Sacks Forcing $\mathbb{S}_\kappa$ and the κ-support product $\mathbb{S}_\kappa^\lambda$.
6. Define preprocessed conditions and the outer hull.
7. Assume by contradiction that $\mathcal{A}$ is not densely maximal in $V^{\mathbb{S}_\kappa^\lambda}$. Let X be a witness to the violation of (4) and $p \in \mathbb{S}(\kappa)$ forcing that.
8. Use the "similar to selective" properties and outer hulls to find $q \leq p$ forcing the opposite of (4).

Construction of V

- Let $\mathbb{P}_{\mathcal{U}} := \{(\mathcal{B}, B) : \mathcal{B}$ is an independent family of size κ, $B \in \mathcal{U}$ and $|\mathcal{B}^h \cap B| = \kappa$ for all $h \in FF(\mathcal{B})\}$.

$(\mathcal{A}, A) \leq_{\mathbb{P}_{\mathcal{U}}} (\mathcal{B}, B)$ if $\mathcal{A} \supseteq \mathcal{B}$

$A \subseteq^* B$

$\mathbb{P}_{\mathcal{U}}$ κ^{++}-cc

$\mathbb{P}_{\mathcal{U}}$ κ^{+}-closed

Construction of V

- Let $\mathbb{P}_{\mathcal{U}} := \{(\mathcal{B}, B) : \mathcal{B}$ is an independent family of size $\kappa, B \in \mathcal{U}$ and $|\mathcal{B}^h \cap B| = \kappa$ for all $h \in FF(\mathcal{B})\}$.
- $\mathbb{P}_{\mathcal{U}}$ is κ-closed and κ^{++}-cc.
- For G a $\mathbb{P}_{\mathcal{U}}$-generic filter define $\mathcal{A} := \bigcup\{\mathcal{B} : \exists B \in \mathcal{U} \text{ s.t } (\mathcal{B}, B) \in G\}$.
- Then $\mathcal{A}$ is an independent family!

Properties of $\mathcal{A}$

Definition

- The *density ideal* of $\mathcal{A}$ is the set $id(\mathcal{A}) := \{X \in \mathcal{U}^* \mid \forall h \in FF(\mathcal{A}) \exists h' \supseteq h \text{ s.t } \mathcal{A}^{h'} \cap X = \emptyset\}$.
- The *density filter* $fil(\mathcal{A})$ is the dual filter of $id(\mathcal{A})$.

Properties of $\mathcal{A}$

Definition

- The *density ideal* of $\mathcal{A}$ is the set $id(\mathcal{A}) := \{X \in \mathcal{U}^* \mid \forall h \in FF(\mathcal{A}) \exists h' \supseteq h \text{ s.t } \mathcal{A}^{h'} \cap X = \emptyset\}$.
- The *density filter* $fil(\mathcal{A})$ is the dual filter of $id(\mathcal{A})$.

Equivalent condition

An independent family $\mathcal{A}$ is densely maximal if and only if for all $h \in FF(\mathcal{A})$ and all $X \subseteq \mathcal{A}^h$

1. Either there is $B \in id(\mathcal{A})$, such that $\mathcal{A}^h \setminus X \subseteq B$
2. or there is $h' \supseteq h$ such that $\mathcal{A}^{h'} \subseteq \mathcal{A}^h \setminus X$.

Properties of $\mathcal{A}$

Definition

- The *density ideal* of $\mathcal{A}$ is the set $id(\mathcal{A}) := \{X \in \mathcal{U}^* \mid \forall h \in FF(\mathcal{A}) \exists h' \supseteq h \text{ s.t } \mathcal{A}^{h'} \cap X = \emptyset\}$.
- The *density filter* $fil(\mathcal{A})$ is the dual filter of $id(\mathcal{A})$.

Equivalent condition

An independent family $\mathcal{A}$ is densely maximal if and only if for all $h \in FF(\mathcal{A})$ and all $X \subseteq \mathcal{A}^h$

1. Either there is $B \in id(\mathcal{A})$, such that $\mathcal{A}^h \setminus X \subseteq B$
2. or there is $h' \supseteq h$ such that $\mathcal{A}^{h'} \subseteq \mathcal{A}^h \setminus X$.

First success

$\mathcal{A}$ is densely maximal!

Properties of $\mathcal{A}$

Definition

A filter $\mathcal{F}$ is a κ-p-set if for every $\mathcal{H} \subseteq \mathcal{F}$ of size $\leq \kappa$ there is $F \in \mathcal{F}$ such that $F \subseteq^* H$ for all $H \in \mathcal{H}$.

modulo $< \kappa$

Properties of $\mathcal{A}$

Definition

A filter $\mathcal{F}$ is a κ-p-set if for every $\mathcal{H} \subseteq \mathcal{F}$ of size $\leq \kappa$ there is $F \in \mathcal{F}$ such that $F \subseteq^* H$ for all $H \in \mathcal{H}$.

Next property

$fil(\mathcal{A})$ is a κ-p-set!

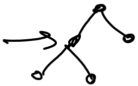
Difficult part

For every $f : \kappa \to \kappa$ which is strictly increasing there is $A \in fil(\mathcal{A})$ with increasing enumeration $\{a_n\}_{n<\kappa}$, such that $f(a_n) < a_{n+1}$.

$\{\xi < \kappa \mid \forall \zeta < \xi \ f(\zeta) < \xi\} \in \mathcal{U}$

Definition

- A set $T \subseteq {}^{<\kappa}2$ which is closed under initial segments is called a *tree*.
- A string $s \in T$ *splits in* T if both $s^\frown 0$ and $s^\frown 1$ are in T.
- A tree T is called *perfect* if for every $s \in T$ there is $s' \supseteq s$ which splits in T and if $(s_n)_{n<\alpha}$ is a sequence of splitting nodes, then there is $s \supseteq s_n$ which splits in T. limit of splitting nodes splits
- The κ-Sacks forcing $\mathbb{S}_\kappa$ consists of all perfect trees with $p \leq_{\mathbb{S}_\kappa} q$ iff $p \subseteq q$.
- $\mathbb{S}^\lambda_\kappa$ is the κ-support product of λ many copies of $\mathbb{S}_\kappa$.



measurable $\Rightarrow$ inacc + $\Diamond_\kappa$ ✓

The Sacks forcing

Definition

- A set $T \subseteq {}^{<\kappa}2$ which is closed under initial segments is called a *tree*.
- A string $s \in T$ *splits in* T if both $s^\frown 0$ and $s^\frown 1$ are in T.
- A tree T is called *perfect* if for every $s \in T$ there is $s' \supseteq s$ which splits in T and if $(s_n)_{n<\alpha}$ is a sequence of splitting nodes, then there is $s \supseteq s_n$ which splits in T.
- The κ-Sacks forcing $\mathbb{S}_\kappa$ consists of all perfect trees with $p \leq_{\mathbb{S}_\kappa} q$ iff $p \subseteq q$.
- $\mathbb{S}_\kappa^\lambda$ is the κ-support product of λ many copies of $\mathbb{S}_\kappa$.

Caution

We need κ to be inaccessible in order to not collapse cardinals $\geq \kappa^{++}$. Also both $\mathbb{S}_\kappa$ and $\mathbb{S}_\kappa^\lambda$ are not κ-closed! $\implies$ Use *fusion orderings*!

Preprocessed conditions and the outer hull

Definition

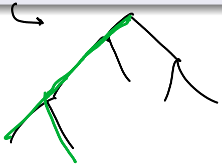
- For a condition $p \in \mathbb{S}_\kappa$ let $split_\alpha(p)$ be the set of all splitting nodes in the α-th splitting level of p.
- For $p \in \mathbb{S}_\kappa^\lambda$, $F \in supp(p)$, $|F| < \kappa$ let $\Lambda_\alpha^F := \{\overline{\sigma} = (\sigma_i) : i \in F, \sigma_i \in split_\alpha(p_i)\}$.
- Let $\dot{X}$ be an $\mathbb{S}_\kappa^\lambda$-name for a set, $p \in S_\kappa^\lambda$, $F \subseteq supp(p)$ with $|F| < \kappa$. Then p is preprocessed for $(F, \dot{X})$ if for every $\overline{\sigma} \in \Lambda_\alpha^F(p)$ there are $F' \subseteq supp(p)$, $|F'| < \kappa$, $F' \supseteq F$, $x \in {}^\alpha 2$ and $\overline{\tau} \supseteq \overline{\sigma}$ with $\overline{\tau} \in \Lambda_\alpha^{F'}(p)$, such that $p_{\overline{\tau}} \Vdash \chi_{\dot{X}} \restriction \alpha = \check{x}$
- p is preprocessed for $\dot{X}$ if it is preprocessed for every pair $(F, \dot{X})$.

$\dot{X}$

Preprocessed conditions and the outer hull

Definition

Let $\dot{X}$ be a name for a set, p preprocessed for $\dot{X}$ and $\overline{\sigma} \in \Lambda_\alpha^F$ for some suitable α, F. Then *the outer hull* of $\dot{X}$ *below* $p_{\overline{\sigma}}$ is the set $Y_{\overline{\sigma}} := \{\beta \in \kappa \mid \exists \tau \supseteq \sigma \text{ s.t } p_{\overline{\tau}} \Vdash \check{\beta} \in \dot{X}\}$.



Preprocessed conditions and the outer hull

Definition

Let $\dot{X}$ be a name for a set, p preprocessed for $\dot{X}$ and $\overline{\sigma} \in \Lambda_\alpha^F$ for some suitable α, F. Then *the outer hull* of $\dot{X}$ *below* $p_{\overline{\sigma}}$ is the set $Y_{\overline{\sigma}} := \{\beta \in \kappa \mid \exists \tau \supseteq \sigma \text{ s.t } p_{\overline{\tau}} \Vdash \check{\beta} \in \dot{X}\}$.

Observation

One can choose $\overline{\tau} \in \Lambda_{\alpha'}^{F'}(p)$ for some α' which is bounded depending on α and β. Also note that $|\Lambda_\alpha^F(p)| < \kappa$ for all α, F.

Proof of $con(\mathfrak{i}(\kappa) < 2^\kappa)$

Claim

$\mathcal{A}$ is densely maximal in $V^{\mathbb{S}_\kappa^\lambda}$.

Proof by contradiction

Assume not. Then there is $h \in FF(\mathcal{A})$ and $X \subseteq \mathcal{A}^h$, such that there is no $B \in id(\mathcal{A})$ with $\mathcal{A}^h \setminus X \subseteq B$, and $\forall h' \supseteq h$ it holds that $\mathcal{A}^{h'} \cap X \neq \emptyset$.

Forcing theorem

Fix such h, let $\dot{X}$ be a name for such a set and $p \in \mathbb{S}_\kappa^\lambda$ **preprocessed** forcing the statement above.

Proof of $con(\mathfrak{i}(\kappa) < 2^\kappa)$

Goal

Want to find $q \leq p$ and $C \in fil(\mathcal{A})$, such that $q \Vdash \check{C} \subseteq \dot{X}$.

This leads to a contradiction because then
$\mathcal{A}^h \setminus X \subseteq \mathcal{A}^h \setminus C \subseteq \kappa \setminus C \in id(\mathcal{A})$.

Proof of $con(\mathfrak{i}(\kappa) < 2^\kappa)$

Goal

Want to find $q \leq p$ and $C \in fil(\mathcal{A})$, such that $q \Vdash \check{C} \subseteq \dot{X}$.

This leads to a contradiction because then
$\mathcal{A}^h \setminus X \subseteq \mathcal{A}^h \setminus C \subseteq \kappa \setminus C \in id(\mathcal{A})$.

How to find such a set and condition?

- Use outer hulls, p-set and "nearly q-set" property
- Enumerate $C = \{c_\alpha\}_{\alpha<\kappa}$
- Have a fusion sequence $(q_\alpha)_{\alpha<\kappa}$, such that $q_{\alpha+1} \Vdash \check{c}_\alpha \in \dot{X}$
- There is q, such that $q \leq q_\alpha$ for all $\alpha < \kappa$ as desired. $\square$

$\lightning$

Some Sources

[1] V. Fischer and D. C. Montoya, “Higher independence,” *Journal of Symbolic Logic*, vol. 87, no. 4, pp. 1606–1630, 2022. DOI: 10.1017/jsl.2022.33.

[2] A. Kanamori, “Perfect-set forcing for uncountable cardinals,” *Annals of Mathematical Logic*, vol. 19, no. 1-2, pp. 97–114, 1980. DOI: 10.1016/0003-4843(80)90021-2.

[3] S. Shelah, “Con(u>i),” *Archive for Mathematical Logic*, vol. 31, no. 6, pp. 433–443, 1992. DOI: 10.1007/bf01277485.

[4] L. K. Gregor Michailowitsch Fichtenholz, “Sur les opérations linéaires dans l’espace des fonctions bornées,” *Studia Mathematica*, vol. 5, pp. 69–98, 1935.

Thanks for your Attention!

Questions?