Determinacy of Long Games

Isabel Macenka and Allison Wang

University of Cambridge

8 April 2021
The Axiom of Determinacy (AD) states that all games of length $\omega$ with move set $\omega$ are determined.
Introduction

The Axiom of Determinacy (AD) states that all games of length $\omega$ with move set $\omega$ are determined.

We will be discussing determinacy axioms for long games — statements about games with move set $\omega$ played on countable ordinals greater than $\omega$. 
Introduction

The Axiom of Determinacy (AD) states that all games of length $\omega$ with move set $\omega$ are determined.

We will be discussing determinacy axioms for long games — statements about games with move set $\omega$ played on countable ordinals greater than $\omega$.

In addition, we will discuss the relation between some of these determinacy axioms and the choice principle $\text{AC}_R(\mathbb{R})$. 
Determinacy and $AC^R(\mathbb{R})$

\[
\begin{align*}
AD^R & \iff AD^{\omega^2} \implies AD^{\omega \cdot 2} \implies AD^{\omega + n} \iff AD \\
& \downarrow \\
& AC^R(\mathbb{R})
\end{align*}
\]
Table of Contents

1 Notation and Definitions

2 Determinacy of games of length $\leq \omega^2$

3 $\text{AD}_R \iff \text{AD}^{\omega^2}$

4 $\text{AC}_R(R)$ and $L(R)$

5 Games on countable ordinals
Notation

We will write $\mathbb{R}$ for $\omega^\omega$ and refer to elements of $\mathbb{R} = \omega^\omega$ as *reals*.
We will write $\mathbb{R}$ for $\omega^\omega$ and refer to elements of $\mathbb{R} = \omega^\omega$ as *reals*. For $x \in X^\alpha$ and $y \in X$, we will write $x \triangleleft y$ for the concatenation of $x$ and $y$. 
Notation

We will write $\mathbb{R}$ for $\omega^\omega$ and refer to elements of $\mathbb{R} = \omega^\omega$ as *reals*.

For $x \in X^\alpha$ and $y \in X$, we will write $x \sqcup y$ for the concatenation of $x$ and $y$.

**Choice principle** $\text{AC}_X(Y)$

(AC$_X(Y)$) For every family $\{A_x \mid x \in X\}$ of non-empty sets $A_x \subseteq Y$, there is a choice function $c : X \to Y$ such that $c(x) \in A_x$ for all $x \in X$. 
An \(\alpha\)-game (or a \textit{game of length} \(\alpha\)) on a non-empty set \(X\) with payoff set \(A\) is played as follows:

1. On turn 0, player I plays some \(x_0 \in X\).
2. Player I and player II take turns playing elements in \(X\).
3. For each limit ordinal \(\lambda < \alpha\), player I plays on turn \(\lambda\).
4. Define \(x \in X^\alpha\) by \(x(\beta) := x_\beta\) for all \(\beta < \alpha\). We call \(x\) a \textit{run} of the game.
5. Player I wins the run \(x\) iff \(x \in A\).
An $\alpha$-game (or a game of length $\alpha$) on a non-empty set $X$ with payoff set $A$ is played as follows:

- On turn 0, player I plays some $x_0 \in X$. 

Define $x \in X^{\alpha}$ by $x(\beta) := x_{\beta}$ for all $\beta < \alpha$. We call $x$ a run of the game.

Player I wins the run $x$ iff $x \in A$. 
Games

An $\alpha$-game (or a game of length $\alpha$) on a non-empty set $X$ with payoff set $A$ is played as follows:

- On turn 0, player I plays some $x_0 \in X$.
- Player I and player II take turns playing elements in $X$.

Define $x \in X^\alpha$ by $x(\beta) := x_\beta$ for all $\beta < \alpha$. We call $x$ a run of the game.

Player I wins the run $x$ iff $x \in A$.
Games

An $\alpha$-game (or a game of length $\alpha$) on a non-empty set $X$ with payoff set $A$ is played as follows:

- On turn 0, player I plays some $x_0 \in X$.
- Player I and player II take turns playing elements in $X$.
- For each limit ordinal $\lambda < \alpha$, player I plays on turn $\lambda$. 

Define $x(\beta) := x_{\beta}$ for all $\beta < \alpha$. We call $x$ a run of the game.

Player I wins the run $x$ iff $x \in A$. 

Macenka and Wang
Determinacy of Long Games
8 April 2021 6 / 26
An $\alpha$-game (or a game of length $\alpha$) on a non-empty set $X$ with payoff set $A$ is played as follows:

- On turn 0, player I plays some $x_0 \in X$.
- Player I and player II take turns playing elements in $X$.
- For each limit ordinal $\lambda < \alpha$, player I plays on turn $\lambda$.

Define $x \in X^\alpha$ by $x(\beta) := x_\beta$ for all $\beta < \alpha$. We call $x$ a run of the game. Player I wins the run $x$ iff $x \in A$. 
Determinacy

A strategy for this game is a function \( \sigma : X^{<\alpha} \rightarrow X \).
A strategy for this game is a function \( \sigma : X^{<\alpha} \rightarrow X \). If players I and II play according to strategies \( \sigma \) and \( \tau \), respectively, let \( \sigma \ast \tau \) denote the run of the game that results.
Determinacy

A strategy for this game is a function $\sigma : X^{<\alpha} \to X$. If players I and II play according to strategies $\sigma$ and $\tau$, respectively, let $\sigma \ast \tau$ denote the run of the game that results. We say $\sigma$ is a winning strategy for player I if $\sigma \ast \tau \in A$ for all strategies $\tau$, and similarly for player II.
A strategy for this game is a function $\sigma : \mathcal{X}^{<\alpha} \to \mathcal{X}$. If players I and II play according to strategies $\sigma$ and $\tau$, respectively, let $\sigma \ast \tau$ denote the run of the game that results. We say $\sigma$ is a winning strategy for player I if $\sigma \ast \tau \in A$ for all strategies $\tau$, and similarly for player II.

Clearly, at most one player has a winning strategy for any game. However, it need not be the case that either player has a winning strategy.

**Determinacy axiom for $\alpha$-games**

($\text{AD}_{\mathcal{X}}^\alpha$) Every $\alpha$-game on the move set $\mathcal{X}$ is determined.
Table of Contents

1 Notation and Definitions

2 Determinacy of games of length $\leq \omega^2$

3 $\text{AD}_R \iff \text{AD}^{\omega^2}$

4 $\text{AC}_R(R)$ and $L(R)$

5 Games on countable ordinals
Zermelo’s Theorem

Theorem

All finite games are determined.
Zermelo’s Theorem

Theorem
All finite games are determined.

Proof Sketch: Consider the tree of all possible runs of the game. Starting from the $n^{th}$ level, label the nodes according to who will win from that position using reverse induction.
$\text{AD}^\alpha$ for $\alpha \leq \omega^2$

**Proposition**

If $\alpha < \beta$, then $\text{AD}^\beta \implies \text{AD}^\alpha$
AD$^\alpha$ for $\alpha \leq \omega^2$

**Proposition**

If $\alpha < \beta$, then $AD^\beta \implies AD^\alpha$

**Proof Sketch:** Assume $AD^\beta$. Let $A \subseteq \omega^\alpha$. Want to show $G(A)$ is determined.
AD\(^\alpha\) for \(\alpha \leq \omega^2\)

**Proposition**

If \(\alpha < \beta\), then \(\text{AD}^\beta \implies \text{AD}^\alpha\)

**Proof Sketch:** Assume \(\text{AD}^\beta\). Let \(A \subseteq \omega^\alpha\). Want to show \(G(A)\) is determined.
Define \(A' \subseteq \omega^\beta\) as follows:

\[
A' = \{ x \in \omega^\beta \mid x \upharpoonright \alpha \in A \}
\]
Proposition

If $\alpha < \beta$, then $\text{AD}^\beta \implies \text{AD}^\alpha$

Proof Sketch: Assume $\text{AD}^\beta$. Let $A \subseteq \omega^\alpha$. Want to show $G(A)$ is determined.

Define $A' \subseteq \omega^\beta$ as follows:

$$A' = \{ x \in \omega^\beta \mid x \upharpoonright \alpha \in A \}$$

Player I (or II) wins the $\alpha$–game $G(A)$ iff Player I (or II) wins the $\beta$–game $G(A')$. Thus $G(A)$ determined.
$\AD^{\omega + n}$ for $n \in \omega$

**Proposition**

$$\AD \implies \AD^{\omega + n}$$
AD^{\omega+n} \text{ for } n \in \omega

Proposition

\[ AD \implies AD^{\omega+n} \]

Proof Sketch: Any game $G(A)$ of length $\omega + n$ can be thought of as two games played back to back: one of length $\omega$, one of length $n$. 
\( \text{AD}^{\omega+n} \) for \( n \in \omega \)

**Proposition**

\[
\text{AD} \implies \text{AD}^{\omega+n}
\]

**Proof Sketch:** Any game \( G(A) \) of length \( \omega + n \) can be thought of as two games played back to back: one of length \( \omega \), one of length \( n \). \( AD \) says the \( \omega \)-game is determined. *Zermelo* says the finite game is determined. Therefore \( G(A) \) is determined.
Table of Contents

1 Notation and Definitions

2 Determinacy of games of length $\leq \omega^2$

3 $\text{AD}_R \iff \text{AD}^{\omega^2}$

4 $\text{AC}_R(\mathbb{R})$ and $L(\mathbb{R})$

5 Games on countable ordinals
Blass’s theorem

Theorem

\[ \text{AD}_R \iff \text{AD}^{\omega^2}. \]
Blass’s theorem

Theorem

$\text{AD}_R \iff \text{AD}^{\omega^2}$.

We will use the following lemmas without proof:
Blass’s theorem

**Theorem**

\[ \text{AD}_\mathbb{R} \iff \text{AD}^{\omega^2}. \]

We will use the following lemmas without proof:

**Lemma 1**

For any set \( X \), \( \text{AC}_X(X) \iff \text{AD}^2_X. \)
Blass’s theorem

**Theorem**

\[ \text{AD}_R \iff \text{AD}^{\omega^2}. \]

We will use the following lemmas without proof:

**Lemma 1**

For any set \( X \), \( \text{AC}_X(X) \iff \text{AD}_X^{2}. \)

**Lemma 2**

For any ordinal \( \alpha \), \( \text{AD}^{\omega \cdot \alpha} \implies \text{AD}_R^{\alpha}. \)
Proof sketch

Theorem

\[ \text{AD}_R \iff \text{AD}^{\omega^2}. \]

Proof sketch: Assume \( \text{AD}_R \). Let \( G \) be an arbitrary \( \omega^2 \)-game on \( \omega \). We think of \( G \) as being composed of \( \omega \)-many “blocks.”
Proof sketch

**Theorem**

\[ \text{AD}_R \iff \text{AD}^{\omega^2} \]

**Proof sketch:** Assume \( \text{AD}_R \). Let \( G \) be an arbitrary \( \omega^2 \)-game on \( \omega \). We think of \( G \) as being composed of \( \omega \)-many “blocks.” Define the \( \omega \)-game \( G' \) on \( \mathbb{R} \) as follows:
Proof sketch

Theorem

$\text{AD}_\mathbb{R} \iff \text{AD}^{\omega^2}$.

Proof sketch: Assume $\text{AD}_\mathbb{R}$. Let $G$ be an arbitrary $\omega^2$-game on $\omega$. We think of $G$ as being composed of $\omega$-many “blocks.” Define the $\omega$-game $G'$ on $\mathbb{R}$ as follows:

<table>
<thead>
<tr>
<th>I</th>
<th>$\sigma_0$</th>
<th>$\sigma_1$</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>II</td>
<td>$q_0$</td>
<td>$q_1$</td>
<td>...</td>
</tr>
</tbody>
</table>

We can think of the move $\sigma_n$ in $G'$ as a strategy $\omega^{<\omega} \rightarrow \omega$ for player I in block $n$ of $G$ by fixing a bijection between $\omega$ and $\omega^{<\omega}$. 
Proof sketch

**Theorem**

\[ \text{AD}_R \iff \text{AD}^{\omega^2}. \]

Proof sketch: Assume \( \text{AD}_R \). Let \( G \) be an arbitrary \( \omega^2 \)-game on \( \omega \). We think of \( G \) as being composed of \( \omega \)-many “blocks.” Define the \( \omega \)-game \( G' \) on \( \mathbb{R} \) as follows:

\[
\begin{array}{c|ccc}
\text{I} & \sigma_0 & \sigma_1 & \ldots \\
\hline
\text{II} & q_0 & q_1 & \ldots \\
\end{array}
\]

We can think of the move \( \sigma_n \) in \( G' \) as a strategy \( \omega^{<\omega} \to \omega \) for player I in block \( n \) of \( G \) by fixing a bijection between \( \omega \) and \( \omega^{<\omega} \). We think of \( q_n \) as being a list of player II’s moves in block \( n \) of \( G \).
Theorem
\[ \text{AD}_\mathbb{R} \iff \text{AD}^{\omega^2}. \]

Proof sketch: Assume \( \text{AD}_\mathbb{R} \). Let \( G \) be an arbitrary \( \omega^2 \)-game on \( \omega \). We think of \( G \) as being composed of \( \omega \)-many “blocks.” Define the \( \omega \)-game \( G' \) on \( \mathbb{R} \) as follows:

<table>
<thead>
<tr>
<th>I</th>
<th>( \sigma_0 )</th>
<th>( \sigma_1 )</th>
<th>( \ldots )</th>
</tr>
</thead>
<tbody>
<tr>
<td>II</td>
<td>( q_0 )</td>
<td>( q_1 )</td>
<td>( \ldots )</td>
</tr>
</tbody>
</table>

We can think of the move \( \sigma_n \) in \( G' \) as a strategy \( \omega^{<\omega} \rightarrow \omega \) for player I in block \( n \) of \( G \) by fixing a bijection between \( \omega \) and \( \omega^{<\omega} \). We think of \( q_n \) as being a list of player II’s moves in block \( n \) of \( G \).

Player I wins a run of \( G' \) iff player I wins the corresponding run of \( G \).
Proof sketch

By \( \text{AD}_R \), \( G' \) is determined.
Proof sketch

By $\mathsf{AD}_{\mathbb{R}}$, $G'$ is determined. If player I has a winning strategy in $G'$, then player I has a winning strategy in $G$. 
Proof sketch

By $\text{AD}_{\mathbb{R}}$, $G'$ is determined. If player I has a winning strategy in $G'$, then player I has a winning strategy in $G$.

Suppose player II has a winning strategy $\tau$ in $G'$.
Proof sketch

By AD$_R$, $G'$ is determined. If player I has a winning strategy in $G'$, then player I has a winning strategy in $G$.

Suppose player II has a winning strategy $\tau$ in $G'$. Let $P$ be the set of all positions or runs in $G$ that correspond to positions or runs in $G'$ arising when player II plays according to $\tau$. 
Proof sketch

By $\text{AD}_R$, $G'$ is determined. If player I has a winning strategy in $G'$, then player I has a winning strategy in $G$.

Suppose player II has a winning strategy $\tau$ in $G'$. Let $P$ be the set of all positions or runs in $G$ that correspond to positions or runs in $G'$ arising when player II plays according to $\tau$. We call elements of $P$ possibilities.
Proof sketch

We say that a sequence $\langle \sigma_i \mid i < n \rangle$ leads to $y \in \omega^\omega \cdot n$ if $y$ is the position in $G$ corresponding to player I playing $\langle \sigma_i \mid i < n \rangle$ and player II playing according to $\tau$ in $G'$. 

Observations:

1. For every $y \in \mathcal{P}$ at least one sequence $\langle \sigma_i \rangle$ leads to $y$.

2. If $\langle \sigma_i \mid i < n \rangle$ leads to $y \in \omega^\omega \cdot n$, then any extension $\langle \sigma_i \rangle \langle \sigma_n \rangle$ leads to an extension $y \langle y_n \rangle \in \omega^\omega \cdot (n + 1)$.
Proof sketch

We say that a sequence $\langle \sigma_i \mid i < n \rangle$ leads to $y \in \omega \cdot n$ if $y$ is the position in $G$ corresponding to player I playing $\langle \sigma_i \mid i < n \rangle$ and player II playing according to $\tau$ in $G'$.

Observations:

1. For every $y \in P$ at least one sequence $\langle \sigma_i \rangle$ leads to $y$. 
Proof sketch

We say that a sequence $\langle \sigma_i \mid i < n \rangle$ leads to $y \in \omega^{\omega \cdot n}$ if $y$ is the position in $G$ corresponding to player I playing $\langle \sigma_i \mid i < n \rangle$ and player II playing according to $\tau$ in $G'$.

Observations:

1. For every $y \in P$ at least one sequence $\langle \sigma_i \rangle$ leads to $y$.
2. If $\langle \sigma_i \mid i < n \rangle$ leads to $y \in \omega^{\omega \cdot n}$, then any extension $\langle \sigma_i \mid i < n \rangle \upharpoonright \sigma_n$ leads to an extension $y \upharpoonright y_n \in \omega^{\omega \cdot (n+1)}$. 
Proof sketch

We say that a sequence $\langle \sigma_i \mid i < n \rangle$ leads to $y \in \omega^{\omega \cdot n}$ if $y$ is the position in $G$ corresponding to player I playing $\langle \sigma_i \mid i < n \rangle$ and player II playing according to $\tau$ in $G'$.

Observations:

1. For every $y \in P$ at least one sequence $\langle \sigma_i \rangle$ leads to $y$.
2. If $\langle \sigma_i \mid i < n \rangle$ leads to $y \in \omega^{\omega \cdot n}$, then any extension $\langle \sigma_i \mid i < n \rangle \upharpoonright \sigma_n$ leads to an extension $y \upharpoonright y_n \in \omega^{\omega \cdot (n+1)}$.

For every $\langle \sigma_i \mid i < n \rangle$, $y \in \omega^{\omega \cdot n}$, and $y \upharpoonright y_n \in \omega^{\omega \cdot (n+1)}$ as in observation (2), fix some extension $\langle \sigma_i \mid i < n \rangle \upharpoonright \sigma_n$ leading to $y \upharpoonright y_n$. 
Proof sketch

We say that a sequence $\langle \sigma_i \mid i < n \rangle$ leads to $y \in \omega^{\omega \cdot n}$ if $y$ is the position in $G$ corresponding to player I playing $\langle \sigma_i \mid i < n \rangle$ and player II playing according to $\tau$ in $G'$.

Observations:

1. For every $y \in P$ at least one sequence $\langle \sigma_i \rangle$ leads to $y$.
2. If $\langle \sigma_i \mid i < n \rangle$ leads to $y \in \omega^{\omega \cdot n}$, then any extension $\langle \sigma_i \mid i < n \rangle \overline{\sigma_n}$ leads to an extension $y \overline{y_n} \in \omega^{\omega \cdot (n+1)}$.

For every $\langle \sigma_i \mid i < n \rangle$, $y \in \omega^{\omega \cdot n}$, and $y \overline{y_n} \in \omega^{\omega \cdot (n+1)}$ as in observation (2), fix some extension $\langle \sigma_i \mid i < n \rangle \overline{\sigma_n}$ leading to $y \overline{y_n}$. Call this extension the standard extension.
Proof sketch

Define a partial function $\Sigma$ on the possibilities inductively by:

1. $\Sigma(\emptyset) = \emptyset$
2. If $y \in \omega^{\omega \cdot n}$ and $\Sigma(y) \in \mathbb{R}^n$ is defined, and if $y' \in \omega^{\omega \cdot (n+1)}$ is an extension of $y$ such that some extension of $\Sigma(y)$ leads to $y'$, define $\Sigma(y')$ to be the corresponding standard extension.

Observations:

1. If $\Sigma(y)$ is defined, then $\Sigma(y)$ leads to $y$.
2. If $\Sigma(y)$ is defined and $z = y|_{(\omega \cdot n)}$ for some $n$, then $\Sigma(z)$ is defined and $\Sigma(z) = \Sigma(y)|_n$.
3. If $y \in \omega^{\omega \cdot 2}$ is such that $\Sigma(y|_{(\omega \cdot n)})$ is defined for all $n \in \omega$, then $y$ is a possibility.

So it suffices to find a strategy for player II in $G$ such that any run consistent with this strategy has the property described in observation (3).
Proof sketch

Define a partial function $\Sigma$ on the possibilities inductively by:

1. $\Sigma(\emptyset) = \emptyset$

2. If $y \in \omega^{\omega \cdot n}$ and $\Sigma(y) \in \mathbb{R}^n$ is defined, and if $y' \in \omega^{\omega \cdot (n+1)}$ is an extension of $y$ such that some extension of $\Sigma(y)$ leads to $y'$, define $\Sigma(y')$ to be the corresponding standard extension.

Observations:

1. If $\Sigma(y)$ is defined, then $\Sigma(y)$ leads to $y$. 

Macenka and Wang  
Determinacy of Long Games  
8 April 2021  17 / 26
Proof sketch

Define a partial function \( \Sigma \) on the possibilities inductively by:

1. \( \Sigma(\emptyset) = \emptyset \)
2. If \( y \in \omega^{\omega \cdot n} \) and \( \Sigma(y) \in \mathbb{R}^n \) is defined, and if \( y' \in \omega^{\omega \cdot (n+1)} \) is an extension of \( y \) such that some extension of \( \Sigma(y) \) leads to \( y' \), define \( \Sigma(y') \) to be the corresponding standard extension.

Observations:

1. If \( \Sigma(y) \) is defined, then \( \Sigma(y) \) leads to \( y \).
2. If \( \Sigma(y) \) is defined and \( z = y \upharpoonright (\omega \cdot n) \) for some \( n \), then \( \Sigma(z) \) is defined and \( \Sigma(z) = \Sigma(y) \upharpoonright n \).
Proof sketch

Define a partial function $\Sigma$ on the possibilities inductively by:

1. $\Sigma(\emptyset) = \emptyset$
2. If $y \in \omega^{\omega \cdot n}$ and $\Sigma(y) \in \mathbb{R}^n$ is defined, and if $y' \in \omega^{\omega \cdot (n+1)}$ is an extension of $y$ such that some extension of $\Sigma(y)$ leads to $y'$, define $\Sigma(y')$ to be the corresponding standard extension.

Observations:

1. If $\Sigma(y)$ is defined, then $\Sigma(y)$ leads to $y$.
2. If $\Sigma(y)$ is defined and $z = y \upharpoonright (\omega \cdot n)$ for some $n$, then $\Sigma(z)$ is defined and $\Sigma(z) = \Sigma(y) \upharpoonright n$.
3. If $y \in \omega^{\omega^2}$ is such that $\Sigma(y \upharpoonright (\omega \cdot n))$ is defined for all $n \in \omega$, then $y$ is a possibility.
Proof sketch

Define a partial function $\Sigma$ on the possibilities inductively by:

1. $\Sigma(\emptyset) = \emptyset$
2. If $y \in \omega^{\cdot n}$ and $\Sigma(y) \in \mathbb{R}^n$ is defined, and if $y' \in \omega^{\cdot (n+1)}$ is an extension of $y$ such that some extension of $\Sigma(y)$ leads to $y'$, define $\Sigma(y')$ to be the corresponding standard extension.

Observations:

1. If $\Sigma(y)$ is defined, then $\Sigma(y)$ leads to $y$.
2. If $\Sigma(y)$ is defined and $z = y \upharpoonright (\omega \cdot n)$ for some $n$, then $\Sigma(z)$ is defined and $\Sigma(z) = \Sigma(y) \upharpoonright n$.
3. If $y \in \omega^{\cdot 2}$ is such that $\Sigma(y \upharpoonright (\omega \cdot n))$ is defined for all $n \in \omega$, then $y$ is a possibility.

So it suffices to find a strategy for player II in $G$ such that any run consistent with this strategy has the property described in observation (3).
Proof sketch

Claim

Let \( y \in \omega^{\omega \cdot n} \) be a position in \( G \) such that \( \Sigma(y) \) is defined. Then there exists a strategy \( \tau_y \) for player II on block \( n \) such that \( \Sigma(y') \) is defined for every extension \( y' \in \omega^{\omega \cdot (n+1)} \) of \( y \) that arises from player II playing according to \( \tau_y \) on block \( n \).

We will use this claim without proof.
Proof sketch

Claim

Let \( y \in \omega^{\omega \cdot n} \) be a position in \( G \) such that \( \Sigma(y) \) is defined. Then there exists a strategy \( \tau_y \) for player II on block \( n \) such that \( \Sigma(y') \) is defined for every extension \( y' \in \omega^{\omega \cdot (n+1)} \) of \( y \) that arises from player II playing according to \( \tau_y \) on block \( n \).

We will use this claim without proof.

\( \text{AC}_R(\mathbb{R}) \) allows us to fix one \( \tau_y \) for every such \( y \) simultaneously.
Proof sketch

Claim

Let $y \in \omega^{\omega \cdot n}$ be a position in $G$ such that $\Sigma(y)$ is defined. Then there exists a strategy $\tau_y$ for player II on block $n$ such that $\Sigma(y')$ is defined for every extension $y' \in \omega^{\omega \cdot (n+1)}$ of $y$ that arises from player II playing according to $\tau_y$ on block $n$.

We will use this claim without proof.

$AC_R(\mathbb{R})$ allows us to fix one $\tau_y$ for every such $y$ simultaneously. Since $\Sigma(\emptyset)$ is defined, “gluing” together these $\tau_y$ gives a strategy for player II on $G$. 
Proof sketch

**Claim**

Let $y \in \omega^{\omega \cdot n}$ be a position in $G$ such that $\Sigma(y)$ is defined. Then there exists a strategy $\tau_y$ for player II on block $n$ such that $\Sigma(y')$ is defined for every extension $y' \in \omega^{\omega \cdot (n+1)}$ of $y$ that arises from player II playing according to $\tau_y$ on block $n$.

We will use this claim without proof.

$\text{AC}_R(R)$ allows us to fix one $\tau_y$ for every such $y$ simultaneously. Since $\Sigma(\emptyset)$ is defined, “gluing” together these $\tau_y$ gives a strategy for player II on $G$. Every run $z$ resulting from player II following this strategy in $G$ has the property that $\Sigma(z \upharpoonright (\omega \cdot n))$ is defined for all $n \in \omega$, so by a previous observation, this is a winning strategy for player II. □
Table of Contents

1 Notation and Definitions

2 Determinacy of games of length $\leq \omega^2$

3 $\text{AD}_{\mathbb{R}} \iff \text{AD}^{\omega^2}$

4 $\text{AC}_{\mathbb{R}}(\mathbb{R})$ and $L(\mathbb{R})$

5 Games on countable ordinals
AC$_R(\mathbb{R})$ and Determinacy

**Proposition**

$\text{AD}^{\omega \cdot 2} \implies \text{AC}_R(\mathbb{R})$
AC\(_R(\mathbb{R})\) and Determinacy

**Proposition**

\[ \text{AD}^{\omega \cdot 2} \implies \text{AC}_{\mathbb{R}}(\mathbb{R}) \]

Will use the following lemmas (same as for Blass’s Theorem) without proof:
AC\(_R(\mathbb{R})\) and Determinacy

**Proposition**

\[ \text{AD}^{\omega \cdot 2} \implies AC_{\mathbb{R}}(\mathbb{R}) \]

Will use the following lemmas (same as for Blass’s Theorem) without proof:

**Lemma 1**

For any set \( X \), \( AC_X(X) \iff AD^2_X \).
AC^R(\mathbb{R}) and Determinacy

Proposition

\[ \text{AD}^{\omega \cdot 2} \implies \text{AC}^R(\mathbb{R}) \]

Will use the following lemmas (same as for Blass’s Theorem) without proof:

Lemma 1

For any set \( X \), \( \text{AC}_X(X) \iff \text{AD}_X^2 \).

Lemma 2

For any ordinal \( \alpha \), \( \text{AD}^{\omega \cdot \alpha} \implies \text{AD}_R^\alpha \).
AC$_R$($\mathbb{R}$) and Determinacy

**Proposition**

\[ AD^{\omega \cdot 2} \implies AC_R(\mathbb{R}) \]

Will use the following lemmas (same as for Blass’s Theorem) without proof:

**Lemma 1**

For any set $X$, $AC_X(X) \iff AD_X^2$.

**Lemma 2**

For any ordinal $\alpha$, $AD^{\omega \cdot \alpha} \implies AD^\alpha_R$.

**Proof of Proposition:** From Lemma 2, we have $AD^{\omega \cdot 2} \implies AD_R^2$. Lemma 1 gives us $AD_R^2 \implies AC_R(\mathbb{R})$. 
AD $\iff AC_R(\mathbb{R})$

In order to show $AD \iff AC_R(\mathbb{R})$, we will show that the inner model $L(\mathbb{R})$ is not a model of $AC_R(\mathbb{R})$. 
Construction of $L(\mathbb{R})$

- $L_0(\mathbb{R}) = \text{transitive closure of } \mathbb{R}$
- $L_\lambda(\mathbb{R}) = \bigcup_{\alpha<\lambda} L_\alpha(\mathbb{R})$
- $L_{\lambda+1}(\mathbb{R}) = \{x \mid x \text{ definable over } L_\lambda(\mathbb{R})\}$

$L(\mathbb{R}) = \bigcup L_\lambda(\mathbb{R})$
Construction of $L(\mathbb{R})$

- $L_0(\mathbb{R}) = \text{transitive closure of } \mathbb{R}$
- $L_\lambda(\mathbb{R}) = \bigcup_{\alpha < \lambda} L_\alpha(\mathbb{R})$
- $L_{\lambda+1}(\mathbb{R}) = \{x \mid x \text{ definable over } L_\lambda(\mathbb{R})\}$

$L(\mathbb{R}) = \bigcup L_\lambda(\mathbb{R})$

Properties of $L(\mathbb{R})$

- Smallest inner model containing $\mathbb{R}$ and the ordinals
- Can code every set with a real and an ordinal
Proposition

$L(\mathbb{R})$ does not have $AC_{\mathbb{R}}(\mathbb{R})$. 

Proof:
Consider, for $x \in \mathbb{R}$, $A^x = \{ y | y \text{ not ordinal definable from } x \}$. We know $A^x$ is nonempty. Now assume we have a choice function $f \in L(\mathbb{R})$ for $\{ A^x | x \in \mathbb{R} \}$. As $f \in L(\mathbb{R})$, it must be ordinal definable from some $x_0 \in \mathbb{R}$. However then $f(x_0) \notin A^{x_0}$. Contradiction.
Proposition

$L(R)$ does not have $AC_R(R)$.

Proof: Consider, for $x \in R$, $A_x = \{y \mid y$ not ordinal definable from $x\}$. We know $A_x$ is nonempty.
**Proposition**

$L(\mathbb{R})$ does not have $\text{AC}_R(\mathbb{R})$.

**Proof:** Consider, for $x \in \mathbb{R}$, $A_x = \{y \mid y$ not ordinal definable from $x\}$. We know $A_x$ is nonempty.
Now assume we have a choice function $f \in L(\mathbb{R})$ for $\{A_x \mid x \in \mathbb{R}\}$.
Proposition

$L(\mathbb{R})$ does not have $AC_{\mathbb{R}}(\mathbb{R})$.

Proof: Consider, for $x \in \mathbb{R}$, $A_x = \{ y \mid y \text{ not ordinal definable from } x \}$. We know $A_x$ is nonempty.

Now assume we have a choice function $f \in L(\mathbb{R})$ for $\{A_x \mid x \in \mathbb{R}\}$.

As $f \in L(\mathbb{R})$, it must be ordinal definable from some $x_0 \in \mathbb{R}$. 
**Proposition**

$L(\mathbb{R})$ does not have $AC_{\mathbb{R}}(\mathbb{R})$.

**Proof:** Consider, for $x \in \mathbb{R}$, $A_x = \{y \mid y \text{ not ordinal definable from } x\}$. We know $A_x$ is nonempty.

Now assume we have a choice function $f \in L(\mathbb{R})$ for $\{A_x \mid x \in \mathbb{R}\}$. As $f \in L(\mathbb{R})$, it must be ordinal definable from some $x_0 \in \mathbb{R}$. However then $f(x_0) \not\in A_{x_0}$. Contradiction.
$L(\mathbb{R})$ and dependent choice

**Axiom of Dependent Choice, DC**

For every nonempty set $X$ and entire relation $R$ on $X$ (i.e. for all $a$, there exists $b$ such that $R(a, b)$), there is a function $f : \omega \to X$ such that for all $n$, $R(f(n), f(n + 1))$. 

**Theorem (Kechris)**

Assume ZF $+$ AD $+$ $V = L(\mathbb{R})$ holds. Then DC holds.
### Axiom of Dependent Choice, DC

For every nonempty set $X$ and entire relation $R$ on $X$ (i.e. for all $a$, there exists $b$ such that $R(a, b)$), there is a function $f : \omega \rightarrow X$ such that for all $n$, $R(f(n), f(n + 1))$.

### Theorem (Kechris)

Assume $ZF + AD + V = L(\mathbb{R})$ holds. Then DC holds.
Table of Contents

1 Notation and Definitions

2 Determinacy of games of length $\leq \omega^2$

3 $\text{AD}_\mathbb{R} \iff \text{AD}^{\omega^2}$

4 $\text{AC}_\mathbb{R}(\mathbb{R})$ and $L(\mathbb{R})$

5 Games on countable ordinals
Countable ordinals and determinacy

We’ve discussed $\text{AD}^\alpha$ for $\alpha \leq \omega^2$. What about $\alpha < \omega_1$?
Countable ordinals and determinacy

We’ve discussed $AD^\alpha$ for $\alpha \leq \omega^2$. What about $\alpha < \omega_1$?

**Theorem (Martin, Woodin)**

For all $\alpha < \omega_1$,

1. $AD_R \implies AD^\alpha$
2. $AD^{\omega \cdot 2} \implies AD^\alpha$. 

Since we know $AD^\alpha \implies AD^{\omega \cdot 2}$ if $\omega \cdot 2 \leq \alpha$, we have $AD^{\omega \cdot 2} \iff AD^\alpha$ for $\omega \cdot 2 \leq \alpha < \omega_1$. 

$AD_R \iff AD^{\omega_2} \iff AD^{\omega \cdot 2} \iff AD^\alpha$. 

Countable ordinals and determinacy

We’ve discussed $\text{AD}^\alpha$ for $\alpha \leq \omega^2$. What about $\alpha < \omega_1$?

**Theorem (Martin, Woodin)**

For all $\alpha < \omega_1$,

- $\text{AD}^\mathbb{R} \implies \text{AD}^\alpha$
- $\text{AD}^{\omega \cdot 2} \implies \text{AD}^\alpha$.

Since we know $\text{AD}^\alpha \implies \text{AD}^{\omega \cdot 2}$ if $\omega \cdot 2 \leq \alpha$, we have $\text{AD}^{\omega \cdot 2} \iff \text{AD}^\alpha$ for $\omega \cdot 2 \leq \alpha < \omega_1$. 
We’ve discussed $\text{AD}^\alpha$ for $\alpha \leq \omega^2$. What about $\alpha < \omega_1$?

**Theorem (Martin, Woodin)**

For all $\alpha < \omega_1$,

- $\text{AD}_R \iff \text{AD}^\alpha$
- $\text{AD}^{\omega \cdot 2} \implies \text{AD}^\alpha$.

Since we know $\text{AD}^\alpha \implies \text{AD}^{\omega \cdot 2}$ if $\omega \cdot 2 \leq \alpha$, we have $\text{AD}^{\omega \cdot 2} \iff \text{AD}^\alpha$ for $\omega \cdot 2 \leq \alpha < \omega_1$.

$$\text{AD}_R \iff \text{AD}^{\omega^2} \iff \text{AD}^{\omega \cdot 2} \iff \text{AD}^\alpha$$