Determinacy of Long Games

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Introduction

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We will be discussing determinacy axioms for long games — statements about games with move set ω played on countable ordinals greater than ω.

In addition, we will discuss the relation between some of these determinacy axioms and the choice principle $\mathsf{AC}_{\mathbb{R}}(\mathbb{R})$.

Determinacy and $AC_{\mathbb{R}}(\mathbb{R})$

$$
\begin{array}{ccccccccc}
AD_{\mathbb{R}} & \Longleftrightarrow & AD^{\omega^2} & \Longrightarrow & AD^{\omega\cdot 2} & \Longrightarrow & AD^{\omega+n} & \Longleftrightarrow & AD \\
 & & & & \Downarrow & \not\swarrow & & & \\
 & & & & AC_{\mathbb{R}}(\mathbb{R}) & & & &
\end{array}
$$

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Notation

We will write $\mathbb{R}$ for ω^ω and refer to elements of $\mathbb{R} = \omega^\omega$ as *reals*.

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Choice principle $\mathrm{AC}_X(Y)$

($\mathrm{AC}_X(Y)$) For every family $\{A_x \mid x \in X\}$ of non-empty sets $A_x \subseteq Y$, there is a choice function $c : X \to Y$ such that $c(x) \in A_x$ for all $x \in X$.

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- For each limit ordinal $\lambda < \alpha$, player I plays on turn λ.

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- Player I and player II take turns playing elements in X.
- For each limit ordinal $\lambda < \alpha$, player I plays on turn λ.

Define $x \in X^\alpha$ by $x(\beta) := x_\beta$ for all $\beta < \alpha$. We call x a *run* of the game. Player I wins the run x iff $x \in A$.

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Clearly, at most one player has a winning strategy for any game. However, it need not be the case that either player has a winning strategy.

Determinacy axiom for α-games

(AD_X^α) Every α-game on the move set X is determined.

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Zermelo's Theorem

Theorem

All finite games are determined.

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Proof Sketch: Consider the tree of all possible runs of the game. Starting from the n^{th} level, label the nodes according to who will win from that position using reverse induction.

AD$^\alpha$ for $\alpha \leq \omega^2$

Proposition

If $\alpha < \beta$, then AD$^\beta \implies$ AD$^\alpha$

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Proof Sketch: Assume AD^β. Let $A \subseteq \omega^\alpha$. Want to show $G(A)$ is determined.

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Define $A' \subseteq \omega^\beta$ as follows:

$$A' = \{x \in \omega^\beta \mid x \upharpoonright \alpha \in A\}$$

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Proof Sketch: Assume AD$^\beta$. Let $A \subseteq \omega^\alpha$. Want to show $G(A)$ is determined.
Define $A' \subseteq \omega^\beta$ as follows:

$$A' = \{x \in \omega^\beta \mid x \restriction \alpha \in A\}$$

Player I (or II) wins the $\alpha-$game $G(A)$ iff Player I (or II) wins the $\beta-$game $G(A')$. Thus $G(A)$ determined.

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Proof Sketch: Any game $G(A)$ of length $\omega + n$ can be thought of as two games played back to back: one of length ω, one of length n.

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Proof Sketch: Any game $G(A)$ of length $\omega + n$ can be thought of as two games played back to back: one of length ω, one of length n.
AD says the ω-game is determined. *Zermelo* says the finite game is determined. Therefore $G(A)$ is determined.

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Lemma 2

For any ordinal α, $\mathsf{AD}^{\omega\cdot\alpha} \implies \mathsf{AD}^{\alpha}_{\mathbb{R}}$.

Proof sketch

Theorem

$\mathsf{AD}_{\mathbb{R}} \iff \mathsf{AD}^{\omega^2}$.

Proof sketch: Assume $\mathsf{AD}_{\mathbb{R}}$. Let G be an arbitrary ω^2-game on ω. We think of G as being composed of ω-many "blocks."

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I	σ_0		σ_1		$\ldots$
II		q_0		q_1	$\ldots$

We can think of the move σ_n in G' as a strategy $\omega^{<\omega} \to \omega$ for player I in block n of G by fixing a bijection between ω and $\omega^{<\omega}$.

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Player I wins a run of G' iff player I wins the corresponding run of G.

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We say that a sequence $\langle \sigma_i \mid i < n \rangle$ *leads to* $y \in \omega^{\omega \cdot n}$ if y is the position in G corresponding to player I playing $\langle \sigma_i \mid i < n \rangle$ and player II playing according to τ in G'.

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Observations:

1. For every $y \in P$ at least one sequence $\langle \sigma_i \rangle$ leads to y.

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Observations:

1. For every $y \in P$ at least one sequence $\langle \sigma_i \rangle$ leads to y.
2. If $\langle \sigma_i \mid i < n \rangle$ leads to $y \in \omega^{\omega \cdot n}$, then any extension $\langle \sigma_i \mid i < n \rangle^\frown \sigma_n$ leads to an extension $y^\frown y_n \in \omega^{\omega \cdot (n+1)}$.

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For every $\langle \sigma_i \mid i < n \rangle$, $y \in \omega^{\omega \cdot n}$, and $y^\frown y_n \in \omega^{\omega \cdot (n+1)}$ as in observation (2), fix some extension $\langle \sigma_i \mid i < n \rangle^\frown \sigma_n$ leading to $y^\frown y_n$.

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For every $\langle \sigma_i \mid i < n \rangle$, $y \in \omega^{\omega \cdot n}$, and $y^\frown y_n \in \omega^{\omega \cdot (n+1)}$ as in observation (2), fix some extension $\langle \sigma_i \mid i < n \rangle^\frown \sigma_n$ leading to $y^\frown y_n$. Call this extension the *standard extension*.

Proof sketch

Define a partial function Σ on the possibilities inductively by:

1. $\Sigma(\emptyset) = \emptyset$
2. If $y \in \omega^{\omega \cdot n}$ and $\Sigma(y) \in \mathbb{R}^n$ is defined, and if $y' \in \omega^{\omega \cdot (n+1)}$ is an extension of y such that some extension of $\Sigma(y)$ leads to y', define $\Sigma(y')$ to be the corresponding standard extension.

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Observations:

1. If $\Sigma(y)$ is defined, then $\Sigma(y)$ leads to y.
2. If $\Sigma(y)$ is defined and $z = y \restriction (\omega \cdot n)$ for some n, then $\Sigma(z)$ is defined and $\Sigma(z) = \Sigma(y) \restriction n$.

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3. If $y \in \omega^{\omega^2}$ is such that $\Sigma(y \restriction (\omega \cdot n))$ is defined for all $n \in \omega$, then y is a possibility.

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3. If $y \in \omega^{\omega^2}$ is such that $\Sigma(y \upharpoonright (\omega \cdot n))$ is defined for all $n \in \omega$, then y is a possibility.

So it suffices to find a strategy for player II in G such that any run consistent with this strategy has the property described in observation (3).

Proof sketch

Claim

Let $y \in \omega^{\omega \cdot n}$ be a position in G such that $\Sigma(y)$ is defined. Then there exists a strategy τ_y for player II on block n such that $\Sigma(y')$ is defined for every extension $y' \in \omega^{\omega \cdot (n+1)}$ of y that arises from player II playing according to τ_y on block n.

We will use this claim without proof.

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$AC_{\mathbb{R}}(\mathbb{R})$ allows us to fix one τ_y for every such y simultaneously. Since $\Sigma(\emptyset)$ is defined, "gluing" together these τ_y gives a strategy for player II on G. Every run z resulting from player II following this strategy in G has the property that $\Sigma(z \upharpoonright (\omega \cdot n))$ is defined for all $n \in \omega$, so by a previous observation, this is a winning strategy for player II. □

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$AC_{\mathbb{R}}(\mathbb{R})$ and Determinacy

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$AD^{\omega \cdot 2} \implies AC_{\mathbb{R}}(\mathbb{R})$

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Lemma 2

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Proof of Proposition: From Lemma 2, we have $\mathsf{AD}^{\omega\cdot 2} \implies \mathsf{AD}^2_{\mathbb{R}}$.
Lemma 1 gives us $\mathsf{AD}^2_{\mathbb{R}} \implies \mathsf{AC}_{\mathbb{R}}(\mathbb{R})$

AD $\not\Rightarrow$ AC$_{\mathbb{R}}(\mathbb{R})$

In order to show AD $\not\Rightarrow$ AC$_{\mathbb{R}}(\mathbb{R})$, we will show that the inner model $L(\mathbb{R})$ is not a model of AC$_{\mathbb{R}}(\mathbb{R})$

$L(\mathbb{R})$

Construction of $L(\mathbb{R})$

- $L_0(\mathbb{R})$ = transitive closure of $\mathbb{R}$
- $L_\lambda(\mathbb{R}) = \bigcup_{\alpha<\lambda} L_\alpha(\mathbb{R})$
- $L_{\lambda+1}(\mathbb{R}) = \{x \mid x \text{ definable over } L_\lambda(\mathbb{R})\}$

$L(\mathbb{R}) = \bigcup L_\lambda(\mathbb{R})$

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Properties of $L(\mathbb{R})$

- Smallest inner model containing $\mathbb{R}$ and the ordinals
- Can code every set with a real and an ordinal

$L(\mathbb{R})$ and $AC_{\mathbb{R}}(\mathbb{R})$

Proposition

$L(\mathbb{R})$ does not have $AC_{\mathbb{R}}(\mathbb{R})$.

$L(\mathbb{R})$ and $\mathrm{AC}_{\mathbb{R}}(\mathbb{R})$

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$L(\mathbb{R})$ does not have $\mathrm{AC}_{\mathbb{R}}(\mathbb{R})$.

Proof: Consider, for $x \in \mathbb{R}$, $A_x = \{y \mid y \text{ not ordinal definable from } x\}$. We know A_x is nonempty.

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Proof: Consider, for $x \in \mathbb{R}$, $A_x = \{y \mid y \text{ not ordinal definable from } x\}$. We know A_x is nonempty.

Now assume we have a choice function $f \in L(\mathbb{R})$ for $\{A_x \mid x \in \mathbb{R}\}$.

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Now assume we have a choice function $f \in L(\mathbb{R})$ for $\{A_x \mid x \in \mathbb{R}\}$.
As $f \in L(\mathbb{R})$, it must be ordinal definable from some $x_0 \in \mathbb{R}$. However then $f(x_0) \notin A_{x_0}$. Contradiction.

$L(\mathbb{R})$ and dependent choice

Axiom of Dependent Choice, DC

For every nonempty set X and entire relation R on X
(i.e. for all a, there exists b such that $R(a, b)$),
there is a function $f : \omega \to X$ such that for all n, $R(f(n), f(n+1))$.

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Theorem (Kechris)

Assume $\mathsf{ZF} + \mathsf{AD} + V = L(\mathbb{R})$ holds. Then DC holds.

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Theorem (Martin, Woodin)

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- $\mathsf{AD}^{\omega \cdot 2} \implies \mathsf{AD}^{\alpha}$.

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Since we know $\mathsf{AD}^{\alpha} \implies \mathsf{AD}^{\omega\cdot 2}$ if $\omega \cdot 2 \leq \alpha$, we have $\mathsf{AD}^{\omega\cdot 2} \iff \mathsf{AD}^{\alpha}$ for $\omega \cdot 2 \leq \alpha < \omega_1$.

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- $\mathsf{AD}_{\mathbb{R}} \implies \mathsf{AD}^{\alpha}$
- $\mathsf{AD}^{\omega \cdot 2} \implies \mathsf{AD}^{\alpha}$.

Since we know $\mathsf{AD}^{\alpha} \implies \mathsf{AD}^{\omega \cdot 2}$ if $\omega \cdot 2 \leq \alpha$, we have $\mathsf{AD}^{\omega \cdot 2} \iff \mathsf{AD}^{\alpha}$ for $\omega \cdot 2 \leq \alpha < \omega_1$.

$$\mathsf{AD}_{\mathbb{R}} \iff \mathsf{AD}^{\omega^2} \iff \mathsf{AD}^{\omega \cdot 2} \iff \mathsf{AD}^{\alpha}$$