A Higher Counterpart to Random Forcing

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This talk summarises *"A parallel to the null ideal for inaccessible λ"* (Shelah, 2017) and *"The Higher Cichoń Diagram"* (Baumhauer, Goldstern & Shelah, 2020)

We will compare random forcing of the **classical** Cantor space ${}^{\omega}2$ (the reals) to a similar forcing $\mathbb{Q}_\kappa$ on the **higher** Cantor space ${}^{\kappa}2$ with κ inaccessible, in particular for κ being weakly compact.

Contents

Classical Reals

The topology on ${}^{\omega}2$ is defined by the basis of clopens $\{[s] \mid s \in {}^{<\omega}2\}$, where $[s] = \{x \in {}^{\omega}2 \mid s \subseteq x\}$.

A set $X \subseteq {}^{\omega}2$ is **nowhere dense** if every open O contains an open $U \subseteq O$ with $U \cap X = \varnothing$. A set $X \subseteq {}^{\omega}2$ is **meagre** if it is the countable union of nowhere dense sets. Let $\mathcal{M} \subseteq \mathcal{P}({}^{\omega}2)$ be the set of meagre sets.

Let μ be the **Lebesgue measure**, generated by $\mu([s]) = 2^{-\mathrm{ot}(s)}$ for basic open $[s]$. Let $\mathcal{N} \subseteq \mathcal{P}({}^{\omega}2)$ be the set of Lebesgue **null** sets.

Proposition

$\mathcal{M}$ and $\mathcal{N}$ are $<\omega_1$-complete ideals and contain all singleton sets. The set of meagre Borel sets is cofinal in $\mathcal{M}$ and the set of null Borel sets is cofinal in $\mathcal{N}$. Finally, $\mathcal{M}$ and $\mathcal{N}$ are orthogonal: there exists $A \in \mathcal{M}$ with ${}^{\omega}2 \setminus A \in \mathcal{N}$.

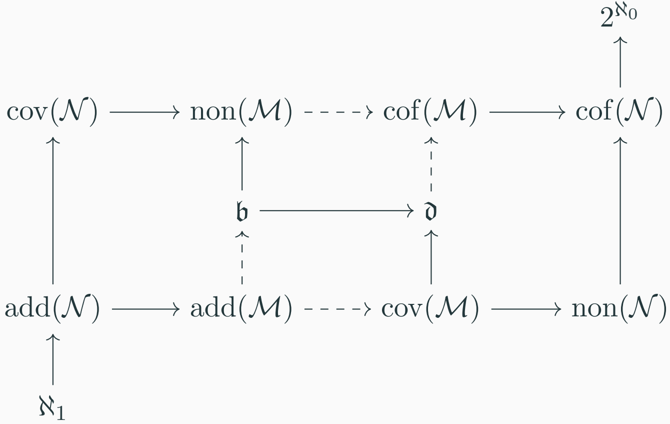
Cardinal characteristics

Let λ be a regular cardinal, $\mathcal{I}$ be a $<\lambda^+$-complete ideal on ${}^{\lambda}2$ containing all singleton sets. We define the following cardinals:

- $\mathrm{cov}(\mathcal{I})$ is the least cardinality of $\mathcal{J} \subseteq \mathcal{I}$ such that $\bigcup \mathcal{J} = {}^{\lambda}2$,
- $\mathrm{non}(\mathcal{I})$ is the least cardinality of $I \subseteq {}^{\lambda}2$ such that $I \notin \mathcal{I}$,
- $\mathrm{add}(\mathcal{I})$ is the least cardinality of $\mathcal{J} \subseteq \mathcal{I}$ such that $\bigcup \mathcal{J} \notin \mathcal{I}$,
- $\mathrm{cof}(\mathcal{I})$ is the least cardinality of $\mathcal{J} \subseteq \mathcal{I}$ such that for all $I \in \mathcal{I}$ there is $J \in \mathcal{J}$ with $I \subseteq J$.

Given $f, g \in {}^{\lambda}\lambda$, let $f \leq^* g$ if there is $\alpha \in \lambda$ such that $f(\xi) \leq g(\xi)$ for all $\xi \geq \alpha$.

- $\mathfrak{b}_\lambda$ is the least cardinality of $B \subseteq {}^{\lambda}\lambda$ such that for all $f \in {}^{\lambda}\lambda$ there is $g \in B$ such that $g \not\leq^* f$,
- $\mathfrak{d}_\lambda$ is the least cardinality of $D \subseteq {}^{\lambda}\lambda$ such that for all $f \in {}^{\lambda}\lambda$ there is $g \in D$ such that $f \leq^* g$.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For any $s \in {}^{<\omega}2$, let $T_s = \{t \in {}^{<\omega}2 \mid s \subseteq t \text{ or } t \subseteq s\}$. Note that T_s is a tree and $T_s \subseteq T_t$ iff $s \supseteq t$.

The **Cohen forcing** $\mathbb{C}$ has as conditions trees T such that $T = T_s$ for some $s \in {}^{<\omega}2$ and is ordered by inclusion: $T' \leq_{\mathbb{C}} T$ iff $T' \subseteq T$, where T' is the **stronger** condition. If $r \in {}^{\omega}2$ is a real added by forcing with $\mathbb{C}$, then r is called a **Cohen real**.

Proposition

Let G be $\mathbb{C}$-generic, then $r \in {}^{\omega}2 \cap \mathbf{V}[G]$ is Cohen iff $r \notin B_c$ for every Borel set $B_c \in \mathcal{M}$ coded by some $c \in {}^{\omega}2 \cap \mathbf{V}$.

Alternatively, $A \in \mathcal{M}$ iff there is a Borel set B_c coded by $c \in {}^{\omega}2$ such that $A \subseteq B_c$ and $\Vdash_{\mathbb{C}}$ "$\dot{r} \notin B_c$", where $\dot{r}$ is the $\mathbb{C}$-name of a generic Cohen real. □

If $T \subseteq {}^{<\omega}2$ is a tree, let $[T]$ be the set of branches of T. Note that $[T]$ is a compact set.

The **random forcing** $\mathbb{R}$ has as conditions trees T such that $\mu([T]) > 0$ and is ordered by inclusion. If $r \in {}^{\omega}2$ is a real added by forcing with $\mathbb{R}$, then r is called a **random real**.

Proposition

Let G be $\mathbb{R}$-generic, then $r \in {}^{\omega}2 \cap \mathbf{V}[G]$ is random iff $r \notin B_c$ for every Borel set $B_c \in \mathcal{N}$ coded by some $c \in {}^{\omega}2 \cap \mathbf{V}$.

Alternatively, $A \in \mathcal{N}$ iff there is a Borel set B_c coded by $c \in {}^{\omega}2$ such that $A \subseteq B_c$ and $\Vdash_{\mathbb{R}}$ "$\dot{r} \notin B_c$", where $\dot{r}$ is the $\mathbb{R}$-name of a generic random real. $\square$

Properties of Classical Random Forcing

A forcing $\mathbb{P}$ is **c.c.c.** if every $\leq_{\mathbb{P}}$-antichain is countable.

A subset $\mathbb{P}' \subseteq \mathbb{P}$ is n-**linked** if every $A \in [\mathbb{P}']^n$ has a lower bound (possibly in $\mathbb{P} \setminus \mathbb{P}'$). $\mathbb{P}$ is σ-n-**linked** if it is the countable union of n-linked sets. $\mathbb{P}$ is σ-**centred** if $\mathbb{P}$ is the countable union of sets that are n-linked for all $n \in \omega$.

$\mathbb{P}$ is ${}^\omega\omega$-**bounding** if for any $\mathbb{P}$-name $\dot{f}$ for a real in ${}^\omega\omega$ there is $g \in {}^\omega\omega$ in the ground model such that $\Vdash_{\mathbb{P}}$ " $\dot{f} \leq^* g$ ".

Proposition

The random forcing $\mathbb{R}$ is c.c.c., σ-n-linked for all $n \in \omega$ and ${}^\omega\omega$-bounding. If $\mathbb{P}$ is a σ-centred forcing, then $\mathbb{P}$ does not add a random real, thus $\mathbb{R}$ is not σ-centred. □

Contents

Let κ be regular uncountable. In analogy to the reals ${}^{\omega}2$, we call elements of ${}^{\kappa}2$ **higher reals**.

For any $s \in {}^{<\kappa}2$ let $[s] = \{x \in {}^{\kappa}2 \mid s \subseteq x\}$. The topology on ${}^{\kappa}2$ is defined by the basis of clopens $\{[s] \mid s \in {}^{<\kappa}2\}$. This is called the $<\kappa$-**box topology**. A set $X \subseteq {}^{\kappa}2$ is **meagre** if it is the union of $\leq\kappa$ nowhere dense sets. Let $\mathcal{M}_\kappa$ be the set of **meagre** sets of ${}^{\kappa}2$.

Proposition

$\mathcal{M}_\kappa$ is a $<\kappa^+$-complete ideal that contains all sets of size $\leq\kappa$. $\square$

The κ-**Cohen forcing** $\mathbb{C}_\kappa$ has as conditions trees $T \subseteq {}^{<\kappa}2$ such that $T = T_s$ for some $s \in {}^{<\kappa}2$ and is ordered by inclusion. If $r \in {}^{\kappa}2$ is a higher real added by forcing with $\mathbb{C}_\kappa$, then r is called a κ-**Cohen real**.

The Ideal $\mathrm{id}(\mathbb{P})$

Let $\mathbb{P}$ be a forcing with conditions being trees on ${}^{\kappa}2$ ordered by inclusion.

- For $J \subseteq \mathbb{P}$ we define $\mathrm{set}_1(J) = \bigcup_{p \in J}[p]$,
  and $\mathrm{set}_0(J) = {}^{\kappa}2 \setminus \mathrm{set}_1(J)$.
- For $\Lambda \subseteq \mathcal{P}(\mathbb{P})$ we define $\mathrm{set}_1(\Lambda) = \bigcap_{J \in \Lambda} \mathrm{set}_1(J)$,
  and $\mathrm{set}_0(\Lambda) = {}^{\kappa}2 \setminus \mathrm{set}_1(\Lambda) = \bigcup_{J \in \Lambda} \mathrm{set}_0(J)$.

Let $A \in \mathrm{id}(\mathbb{P})$ iff $A \subseteq \mathrm{set}_0(\Lambda)$ for $\Lambda \subseteq \mathcal{P}(\mathbb{P})$ with $|\Lambda| \leq \kappa$ and each $J \in \Lambda$ predense in $\mathbb{P}$.

Proposition

$\mathrm{id}(\mathbb{P})$ is a $<\kappa^+$-complete ideal. $\square$

Lemma

$\mathrm{id}(\mathbb{C}_\kappa) = \mathcal{M}_\kappa$.

Proof. If $J \subseteq \mathbb{C}_\kappa$ is predense, then $\mathrm{set}_0(J)$ is nowhere dense. If Λ is a family of predense sets with $|\Lambda| \leq \kappa$, then $\mathrm{set}_0(\Lambda)$ is the κ-union of nowhere dense sets, thus meagre.

If $A = \bigcup_{\alpha<\kappa} A_\alpha$ is meagre, with A_α nowhere dense, then there are open dense sets $B_\alpha \subseteq {}^\kappa 2 \setminus A_\alpha$. For each open dense B_α there is a predense $J \subseteq \mathbb{C}_\kappa$ such that $\mathrm{set}_1(J) = B_\alpha$. □

Generalising Random Forcing

Generally speaking, there is no clear way to generalise Lebesgue measure to ${}^{\kappa}2$. Random forcing is defined using Lebesgue measure, thus there is no clear way to generalise random forcing.

Problem

Assume $\kappa^+ < 2^\kappa$. Is there a nontrivial forcing with conditions being trees on ${}^{<\kappa}2$ that is $<\kappa^+$-c.c., (strategically) $<\kappa$-closed and ${}^{\kappa}\kappa$-bounding?

If κ is weakly compact, the answer is **yes**.

The Forcing $\mathbb{Q}_\kappa$

Let κ be (strongly) inaccessible and let S^κ_{inc} be the set of (strongly) inaccessibles below κ. A set $S \subseteq S^\kappa_{\mathrm{inc}}$ is **nowhere stationary** if $S \cap \alpha$ is nonstationary for every $\alpha \in \{\kappa\} \cup S^\kappa_{\mathrm{inc}}$.

$\mathbb{Q}_\kappa$ is defined by recursion over $\lambda \in S^\kappa_{\mathrm{inc}}$. The conditions of $\mathbb{Q}_\kappa$ are trees $p \subseteq {}^{<\kappa}2$ witnessed by a triple $\langle \tau_p, S_p, \overline{\Lambda}_p \rangle$, where:

- $\tau_p \in p$ is the stem of p,
- $S_p \subseteq S^\kappa_{\mathrm{inc}} \setminus (\mathrm{ot}(\tau_p) + 1)$ is nowhere stationary,
- $\overline{\Lambda}_p = \langle \Lambda^\lambda_p \mid \lambda \in S^\kappa_{\mathrm{inc}} \rangle$ is a sequence where for each $\lambda \in S^\kappa_{\mathrm{inc}}$ with $\mathrm{ot}(\tau_p) < \lambda$ we have a family $\Lambda^\lambda_p \subseteq \mathcal{P}(\mathbb{Q}_\lambda)$ of predense subsets with $|\Lambda^\lambda_p| \leq \lambda$,
- if $s \in {}^\alpha 2$ for $\alpha < \kappa$, then $s \in p$ iff both:
  - $s \restriction \beta \in p$ for all $\beta < \alpha$, and
  - $\alpha \notin S_p$ or [$\alpha \in S_p$ and $s \in \mathrm{set}_1(\Lambda^\alpha_p)$].

The Forcing $\mathbb{Q}_\kappa$

Lemma

If $p, q \in \mathbb{Q}_\kappa$ and $\tau_p \in q$ and $\tau_q \in p$ (in particular if $\tau_p = \tau_q$), then $p \cap q$ is a condition.

Lemma

If $\kappa > \sup(S^\kappa_{\text{inc}})$, then $\mathbb{Q}_\kappa$ is forcing equivalent to $\mathbb{C}_\kappa$ and $\text{id}(\mathbb{Q}_\kappa) = \mathcal{M}_\kappa$.

Proofs. By picture:

□

For this reason we will always assume $\kappa = \sup(S^\kappa_{\text{inc}})$ when κ is mentioned.

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- ○ Higher Cohen Forcing & $\mathbb{Q}_\kappa$ Forcing
- • Properties of $\mathbb{Q}_\kappa$
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Forcing properties

A forcing $\mathbb{P}$ is $<\kappa^+$-**c.c.** if every $\leq_{\mathbb{P}}$-antichain is of cardinality $<\kappa^+$.

A forcing $\mathbb{P}$ is $<\kappa$-**closed** if any sequence $p_0 \geq_{\mathbb{P}} p_1 \geq_{\mathbb{P}} \cdots$ of length $<\kappa$ has a lower bound.

Let $\lambda \leq \kappa$. A subset $\mathbb{P}' \subseteq \mathbb{P}$ is **centred**$_{<\lambda}$ if every $A \in [\mathbb{P}']^{<\lambda}$ has a lower bound (possibly in $\mathbb{P} \setminus \mathbb{P}'$). $\mathbb{P}$ is κ-**centred**$_{<\lambda}$ if it is the κ-union of centred$_{<\lambda}$ sets.

$\mathbb{P}$ is ${}^\kappa\kappa$-**bounding** if for any $\mathbb{P}$-name $\dot{f}$ for a higher real in ${}^\kappa\kappa$ there is $g \in {}^\kappa\kappa$ in the ground model such that $\Vdash_{\mathbb{P}}$ "$\dot{f} \leq^* g$".

Forcing properties

For a forcing $\mathbb{P}$, let $\mathcal{G}_\mathbb{P}$ be the game of length κ with the following rules:

- Black and White alternatingly choose $p_\alpha \in \mathbb{P}$ stronger than all previous moves p_β with $\beta < \alpha$,
- Black plays p_0,
- White plays first at limit stages.

$\mathbb{P}$ is **strategically** $<\kappa$**-closed** if White has a strategy to not run out of moves in the game $\mathcal{G}_\mathbb{P}$.

Proposition

If $\mathbb{P}$ is **(strategically)** $<\kappa$**-closed**, then any set $f \in {}^{<\kappa}2$ in the extension after forcing with $\mathbb{P}$ was already in the ground model. $\square$

Theorem

$\mathbb{Q}_\kappa$ is κ-centred$_{<\lambda}$ for all $\lambda < \kappa$. In particular $\mathbb{Q}_\kappa$ is $<\kappa^+$-c.c.

Proof. For each $\tau \in {}^{<\kappa}2$, let $\mathbb{Q}_\kappa^\tau = \{p \in \mathbb{Q}_\kappa \mid \tau_p = \tau\}$. Clearly $\bigcup_{\mathrm{ot}(\tau)\geq\lambda} \mathbb{Q}_\kappa^\tau$ is dense in $\mathbb{Q}_\kappa$. Consider $\{p_\xi \in \mathbb{Q}_\kappa^\tau \mid \xi < \mu\}$ for some τ with $\mathrm{ot}(\tau) \geq \lambda$ and $\mu < \lambda$, and let p_ξ be witnessed by $\langle \tau, S_\xi, \overline{\Lambda}_\xi \rangle$. Then $S_\mu = \bigcup_{\xi<\mu} S_\xi$ is nowhere stationary, and $\overline{\Lambda}_\mu$ with $\Lambda_\mu^\eta = \bigcup_{\xi<\mu} \Lambda_\xi^\eta$ has $|\Lambda_\mu^\eta| \leq \eta$ for all $\eta \geq \lambda$.

Therefore $p = \bigcap_{\xi<\mu} p_\xi$ is a condition witnessed by $\langle \tau, S_\mu, \overline{\Lambda}_\mu \rangle$ and $p \leq p_\xi$ for all $\xi < \mu$. □

Theorem

If $\mathbb{P}$ is κ-centred$_{<\kappa}$ and preserves ${}^{<\kappa}2$, then $\mathbb{P}$ does not add a $\mathbb{Q}_\kappa$-generic higher real. □

Lemma

$\mathbb{Q}_\kappa$ is not $<\kappa$-closed.

Proof. Let $\alpha = \min(S)$ for some nowhere stationary $S \subseteq S^\kappa_{\mathrm{inc}}$. Let $\langle p_\beta \mid \beta < \alpha \rangle$ be witnessed by $\left\langle \tau \restriction \beta, S, \overline{\Lambda} \right\rangle$ with $\tau \in \mathrm{set}_0(\Lambda^\alpha)$, then this sequence has no lower bound. □

Theorem

$\mathbb{Q}_\kappa$ is strategically $<\kappa$-closed.

Proof sketch. At White's turn β, White chooses a p_β and a club C_β such that for $\xi \leq \beta$ we have $S_\beta = \bigcup_{\xi<\beta} S_\xi \setminus \beta$, $C_\beta \subseteq \bigcap_{\xi<\beta} C_\xi \setminus \beta$ such that $C_\beta \cap S_\beta = \varnothing$ and $\mathrm{ot}(\tau_\beta) \in C_\beta$. □

Corollary

If G is $\mathbb{Q}_\kappa$-generic, then $({}^{<\kappa}2)^{\mathbf{V}} = ({}^{<\kappa}2)^{\mathbf{V}[G]}$. □

Theorem

If κ is weakly compact, then $\mathbb{Q}_\kappa$ is ${}^\kappa\kappa$-bounding.

Proof. Let $\Vdash_{\mathbb{Q}_\kappa}$ " $\dot{f} \in {}^\kappa\kappa$ " for a name $\dot{f}$.

For all $p_0 \in \mathbb{Q}_\kappa$ we want to find $p \leq p_0$ and $\langle \beta_\alpha \mid \alpha < \kappa \rangle \subseteq \kappa$ such that if $r \leq p$ and $\mathrm{ot}(\tau_r) = \beta_{\alpha+1}$, then $r \Vdash$ " $\dot{f}(\alpha) = \eta^r_\alpha$ " for some η^r_α. Then $p \Vdash$ " $\dot{f}(\alpha) \leq \eta_\alpha$ " for η_α greater than all η^r_α. Let $g_p : \alpha \mapsto \eta_\alpha$, then $p \Vdash$ " $\dot{f} \leq g_p$ ".

Let P be dense such that for any p_0 there is $p \in P$ as above. We can find a g_p for each $p \in A \subseteq P$, where A is a maximal antichain. Since $\mathbb{Q}_\kappa$ is $<\!\kappa^+$-c.c., then $\{g_p \mid p \in A\}$ is $\leq^*$-bounded by some g, thus $\Vdash_{\mathbb{Q}_\kappa}$ " $\dot{f} \leq^* g$ ". ...

Cont'd. Let p_0 be witnessed by $\left\langle \tau, S_0, \overline{\Lambda}_0 \right\rangle$, then we find descending $\langle p_\alpha \rangle$ witnessed by $\left\langle \tau, S_\alpha, \overline{\Lambda}_\alpha \right\rangle$ such that if $r \leq p_{\alpha+1}$ and $\mathrm{ot}(\tau_r) = \beta_{\alpha+1}$, then r decides $\dot{f}(\alpha)$. We will let $p = \bigcap_{\alpha<\kappa} p_\alpha$. For $\alpha < \kappa$ we also define a descending chain of clubs C^α with $C^{\alpha+1} \cap S_\alpha = \varnothing$. We take $\beta_0 \in C^0 \subseteq \kappa \setminus (S_0 \cup \mathrm{ot}(\tau))$ arbitrary.

Given p_α, let $\langle q_\xi \mid \xi < \kappa \rangle$ be a maximal antichain below p_α with $q_\xi \Vdash$ "$\dot{f}(\alpha) = \eta_\xi$" for $\eta_\xi < \kappa$. Let S_∇ the diagonal union of $\left\langle S_{q_\xi} \right\rangle$. Take $C^{\alpha+1} \subseteq C^\alpha \setminus (S_\nabla \cup S_\alpha \cup \beta_\alpha)$ and $C^{\alpha+1}_{\mathrm{inc}} = S^\kappa_{\mathrm{inc}} \cap C^{\alpha+1}$ (which is stationary since κ is Mahlo).

Claim

There exists $\lambda_\alpha \in C^{\alpha+1}_{\mathrm{inc}}$ such that $\left\{ q_\xi \cap {}^{<\lambda_\alpha}2 \mid \xi < \lambda_\alpha \right\}$ is a maximal antichain below $p_\alpha \cap {}^{<\lambda_\alpha}2$. ...

Properties of $\mathbb{Q}_\kappa$: ${}^\kappa\kappa$-bounding

Proof of claim. Given $\tau \subseteq \sigma \in p_\alpha$ and $\gamma = \mathrm{ot}(\sigma) + 1$, we define:

$$\mathcal{T}_\sigma = \bigcup_{\lambda \in C_{\mathrm{inc}}^{\alpha+1} \setminus \gamma} \{ r \in \mathbb{Q}_\lambda \mid \tau_r = \sigma \text{ and } r \subseteq p_\alpha \text{ and } \forall \xi < \lambda (r \perp q_\xi \cap {}^{<\lambda}2) \quad \}$$

ordered by $r \preceq r'$ iff $r \in \mathbb{Q}_\lambda$, $r' \in \mathbb{Q}_{\lambda'}$, $\lambda \leq \lambda'$ and $r = r' \cap {}^{<\lambda}2$. Assume $\mathcal{T}_\sigma$ has height κ. Each level of $\mathcal{T}_\sigma$ has size $<\kappa$, so by weak compactness let $\langle r_\lambda \rangle$ be a branch, witnessed by $S_{r_\lambda} \subseteq \lambda$ such that $S_{r_\lambda} \subseteq S_{r_{\lambda'}}$ for $\lambda < \lambda'$. Since κ is a reflecting cardinal, $\bigcup_\lambda S_{r_\lambda}$ is nowhere stationary. Hence $r = \bigcup_\lambda r_\lambda \in \mathbb{Q}_\kappa$ and $r \perp q_\xi$ for all ξ, contradiction. Thus let λ_σ be such that $\mathbb{Q}_{\lambda_\sigma} \cap \mathcal{T}_\sigma = \varnothing$.

Given $\lambda \in C_{\mathrm{inc}}^{\alpha+1}$, let f be a continuous function with $f(0) = \lambda$ and $f(\alpha + 1) = \sup \{ \lambda_\sigma \mid \sigma \in {}^{<f(\alpha)}2 \}$. Then there is $\lambda_\alpha \in C_{\mathrm{inc}}^{\alpha+1}$ that is an f-fixed point. This λ_α satisfies the claim. ∎

Properties of $\mathbb{Q}_\kappa$: ${}^\kappa\kappa$-bounding

Cont'd. Let q_ξ^s be witnessed by $\left\langle s, S_{q_\xi} \setminus (\lambda_\alpha + 1), \overline{\Lambda}_{q_\xi} \right\rangle$ for each $s \in q_\xi \cap {}^{\lambda_\alpha}2$. Let $Q \subseteq \mathbb{Q}_{\lambda_\alpha}$ be predense with $\left\{ q_\xi \cap {}^{<\lambda_\alpha}2 \mid \xi < \lambda_\alpha \right\}$ being the part of Q below p_α.

We set $p_{\alpha+1} = \bigcup \left\{ q_\xi^s \mid \xi < \lambda_\alpha \text{ and } s \in q_\xi \cap {}^{\lambda_\alpha}2 \right\}$, witnessed by:

$$S_{\alpha+1} = (S_\alpha \cap \lambda_\alpha) \cup \{\lambda_\alpha\} \cup \bigcup_{\xi<\lambda_\alpha} S_{q_\xi} \setminus (\lambda_\alpha + 1),$$

$$\overline{\Lambda}_{\alpha+1} = \left\langle \Lambda_{\alpha+1}^\lambda \mid \lambda \in S_{\text{inc}}^\kappa \right\rangle \text{ with}$$

$$\Lambda_{\alpha+1}^\lambda = \begin{cases} \Lambda_\alpha^\lambda & \text{if } \lambda < \lambda_\alpha \\ \{Q\} & \text{if } \lambda = \lambda_\alpha \\ \bigcup_{\xi<\lambda_\alpha} \Lambda_{q_\xi}^\lambda & \text{if } \lambda > \lambda_\alpha \end{cases}$$

Take $\beta_{\alpha+1} \in C^{\alpha+1} \setminus (S_{\alpha+1} \cup \lambda_\alpha)$. Then $r \le p_{\alpha+1}$ with $\mathrm{ot}(\tau_r) = \beta_{\alpha+1} > \lambda_\alpha$ decides $\dot{f}(\alpha)$. ...

Cont'd. Finally for limit γ we set $\beta_\gamma = \bigcup_{\alpha<\gamma} \beta_\alpha$ and $p_\gamma = \bigcap_{\alpha<\gamma} p_\alpha$, witnessed by $S_\gamma = \bigcup_{\alpha<\gamma} S_\alpha$.

We let $C^\gamma = \bigcap_{\alpha<\gamma} C^\alpha$, which is club, then since $C^{\alpha+1}$ is disjoint from S_α for each α, we see C^γ is disjoint from S_γ. We have $\beta_\gamma \in C^\gamma$ because $\{\beta_\xi \mid \xi > \alpha\} \subseteq C^\alpha$ for each $\alpha < \gamma$, thus $\beta_\gamma \notin S_\gamma$.

For $\xi \leq \alpha < \gamma$ we have $\beta_\alpha \notin S_\xi$, and for $\xi > \alpha$ we have $S_\xi \cap \lambda_\alpha = S_\alpha \cap \lambda_\alpha$, therefore by $\beta_\alpha < \lambda_\alpha$ we get $\beta_\alpha \notin S_\xi$. Therefore S_γ is nonstationary below β_γ, and since S_γ is a γ-union of nowhere stationary sets, it is also nonstationary above β_γ. Finally if $\gamma = \beta_\gamma$, then $\langle \beta_\alpha \mid \alpha < \gamma \rangle$ is a club set disjoint from $S_\gamma \cap \gamma$. □

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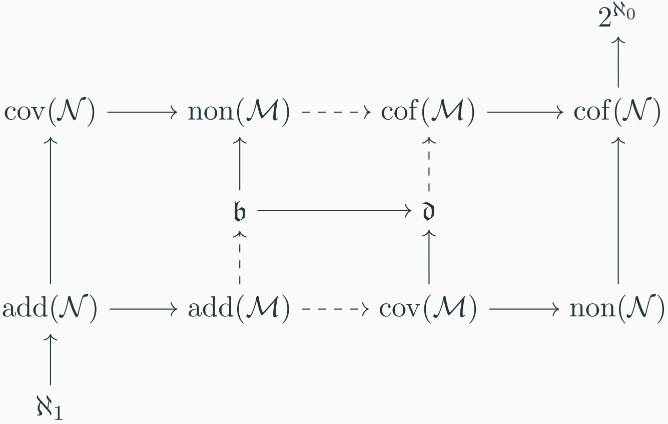
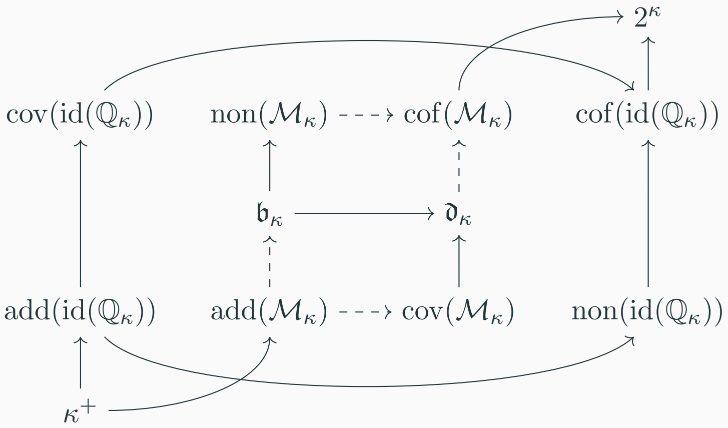
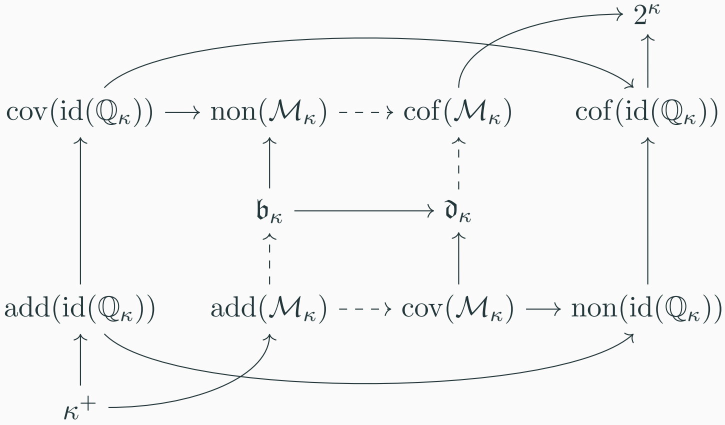
Cichoń's Diagram



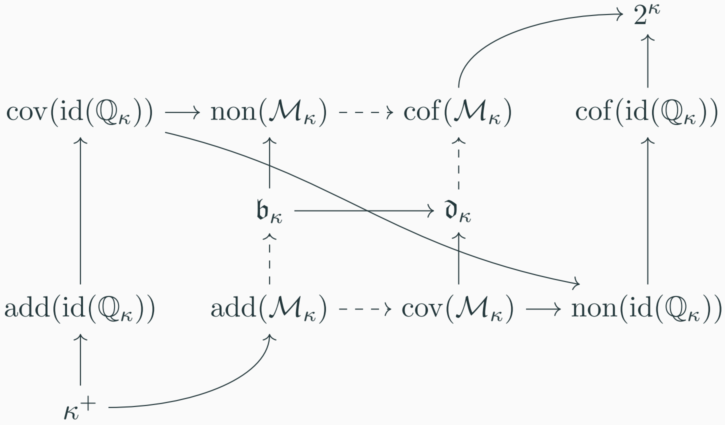
Higher Cichoń's Diagram



Higher Cichoń's Diagram



Higher Cichoń's Diagram



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- • Anti-Fubini Sets & Orthogonality

Let $x \in X$, $y \in Y$ and $A \subseteq X \times Y$, then define the sections $A_x = \{y \in Y \mid (x, y) \in A\}$ and $A^y = \{x \in X \mid (x, y) \in A\}$.

Theorem (Fubini's theorem)

Let $A \subseteq {}^{\omega}2 \times {}^{\omega}2$ be measurable, then $\mu(A) = 0$ iff $\{x \in {}^{\omega}2 \mid \mu(A_x) > 0\} \in \mathcal{N}$ iff $\{y \in {}^{\omega}2 \mid \mu(A^y) > 0\} \in \mathcal{N}$. □

An **anti-Fubini set** between the ideals $\mathcal{I}$ and $\mathcal{J}$ is a set $F \subseteq {}^{\kappa}2 \times {}^{\kappa}2$ for which $F_x \in \mathcal{I}$ and ${}^{\kappa}2 \setminus F^y \in \mathcal{J}$ for all $x, y \in {}^{\kappa}2$.

Lemma

If $A \subseteq {}^{\kappa}2 \times {}^{\kappa}2$ is anti-Fubini between $\mathcal{I}$ and $\mathcal{J}$, then $\mathrm{cov}(\mathcal{I}) \leq \mathrm{non}(\mathcal{J})$ and $\mathrm{cov}(\mathcal{J}) \leq \mathrm{non}(\mathcal{I})$.

Proof. Let F be anti-Fubini, $Y \notin \mathcal{I}$. Since $F_x \in \mathcal{I}$, let $y \in Y \setminus F_x$, then $(x, y) \notin F$, so $x \notin F^y$, so $x \in \bigcup_{y \in Y} {}^{\kappa}2 \setminus F^y$. □

Theorem

There exists an anti-Fubini set between $\mathbb{Q}_\kappa$ and itself.

Proof. Let $S = \{\lambda \mid \lambda > \sup(S^\lambda_{\text{inc}})\}$ and fix $x \in {}^\kappa 2$, $\lambda \in S$. For $s \in {}^{<\lambda}2$ let $x^s_\lambda \in {}^\lambda 2$ be defined as $x^s_\lambda \restriction \operatorname{dom}(s) = s$ and $x^s_\lambda(\alpha) = x(\lambda + \alpha)$ otherwise. Let $A^x_\lambda = \{x^s_\lambda \mid s \in {}^{<\lambda}2\} \in \operatorname{id}(\mathbb{Q}_\lambda)$.

Let $\Lambda^x_\lambda \subseteq \mathcal{P}(\mathbb{Q}_\lambda)$ witness that $A^x_\lambda \subseteq \operatorname{set}_0(\Lambda^x_\lambda)$ for $\lambda \in S$, and define a sequence $\overline{\Lambda}^x = \langle \Lambda^x_\lambda \mid \lambda \in S^\kappa_{\text{inc}} \rangle$. For each $\tau \in {}^{<\kappa}2$ let $p^x_\tau \in \mathbb{Q}_\kappa$ be witnessed by $\langle \tau, S \setminus \operatorname{ot}(\tau), \overline{\Lambda}^x \rangle$. $J^x = \{p^x_\tau \mid \tau \in {}^{<\kappa}2\}$ is predense.

Unfixing x, let $F = \{(x, y) \in {}^\kappa 2 \times {}^\kappa 2 \mid y \in \operatorname{set}_1(\{J^x\})\}$, then $F_x = \operatorname{set}_1(\{J^x\})$, thus ${}^\kappa 2 \setminus F_x = \operatorname{set}_0(\{J^x\}) \in \operatorname{id}(\mathbb{Q}_\kappa)$. We have to show that $F^y \in \operatorname{id}(\mathbb{Q}_\kappa)$ for all $y \in {}^\kappa 2$. …

Cont'd. Let $D_y^\alpha = \left\{p \in \mathbb{Q}_\kappa \mid \exists\lambda \in S \setminus \alpha(y \restriction \lambda \in \bigcap_{x\in[p]} A_\lambda^x)\right\}$ for $y \in {}^\kappa 2$. We show D_y^α is dense and $F^y \subseteq \mathrm{set}_0(\{D_y^\alpha \mid \alpha < \kappa\})$.

Take $q \in \mathbb{Q}_\kappa$, w.l.o.g. with $S_q \setminus \alpha \neq \varnothing$. Let $\lambda \in S_q \setminus \alpha$, then since $(\lambda, \lambda \cdot 2) \cap S_{\mathrm{inc}}^\kappa = \varnothing$, we see that q is fully branching in $[\lambda, \lambda \cdot 2)$. Hence $t' \in q \cap {}^\lambda 2$ and $t'' \in {}^\lambda 2$ implies $t'^\frown t'' \in q$. Let $t \in q \cap {}^{\lambda\cdot 2}2$ with $t \restriction [\lambda, \lambda \cdot 2) = y \restriction [0, \lambda)$. Then let $r \leq q$ be such that $t \subseteq \tau_r$. If $x \in [r]$, then take $s = y \restriction \xi$ for some $\xi < \lambda$, to see that $y \restriction \lambda = x_\lambda^s \in A_\lambda^x$, so $r \in D_y^\alpha$.

Let $x \in F^y$, then $y \in \mathrm{set}_1(\{J^x\})$. Thus $y \in [p_\tau^x]$ for some $\tau \in {}^{<\kappa}2$, where p_τ^x is witnessed by $\left\langle \tau, S \setminus \mathrm{ot}(\tau), \overline{\Lambda}^x \right\rangle$. For each $\lambda \in S \setminus \mathrm{ot}(\tau)$ we see that $y \restriction \lambda \notin \mathrm{set}_0(\Lambda_\lambda^x)$, and thus by $A_\lambda^x \subseteq \mathrm{set}_0(\Lambda_\lambda^x)$ also $y \restriction \lambda \notin A_\lambda^x$. Therefore $x \in \mathrm{set}_0(D_y^\alpha)$ for any $\alpha \geq \mathrm{ot}(\tau)$. $\square$

Orthogonality

Theorem

There exists $A \subseteq {}^{\kappa}2$ such that $A \in \mathcal{M}_\kappa$ and ${}^{\kappa}2 \setminus A \in \mathrm{id}(\mathbb{Q}_\kappa)$.

Proof. Let $S = \left\{\lambda \mid \lambda > \sup(S^{\lambda}_{\mathrm{inc}})\right\}$ and let $\lambda^- = \sup(S^{\lambda}_{\mathrm{inc}})$ and $L_\lambda = \{p \in \mathbb{Q}_\lambda \mid \exists\alpha(\lambda^- \leq \alpha < \mathrm{ot}(\tau_p)$ and $\tau_p(\alpha) \neq 0)\}$. Let $p_\eta \in \mathbb{Q}_\kappa$ be witnessed by $\langle \eta, S \setminus \mathrm{ot}(\eta), \langle \{L_\lambda\} \mid \lambda \in S\rangle\rangle$. Then $\mathrm{set}_1(\{p_\eta \mid \eta \in {}^{<\kappa}2\}) \in \mathcal{M}_\kappa$ and $\mathrm{set}_0(\{p_\eta \mid \eta \in {}^{<\kappa}2\}) \in \mathrm{id}(\mathbb{Q}_\kappa)$. □

Theorem

There exists an anti-Fubini set between $\mathcal{M}_\kappa$ and $\mathrm{id}(\mathbb{Q}_\kappa)$

Proof. Let $A \in \mathcal{M}_\kappa$ be closed under translation such that ${}^{\kappa}2 \setminus A \in \mathrm{id}(\mathbb{Q}_\kappa)$. Define $F = \{(x, y) \in {}^{\kappa}2 \times {}^{\kappa}2 \mid y \in x + A\}$. Then $F_x = x + A \in \mathcal{M}_\kappa$, and thus ${}^{\kappa}2 \setminus F_x = x + {}^{\kappa}2 \setminus A \in \mathrm{id}(\mathbb{Q}_\kappa)$. $F^y = \{x \mid y \in x + A\} = \{x \mid x \in y + -A\} = y + -A \in \mathcal{M}_\kappa$. □

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