Borel chromatic numbers as cardinal invariants

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1. Borel chromatic numbers and other small cardinals
   - The graph counterpart to Vitali's relation
   - A measure-sensitive relative of G_0
2. How to increase Borel chromatic numbers
   - Looking for embeddings of G_0
   - The case with $\chi_B(E_0)$ and $\chi_B(G_1)$
   - The case with $\chi_B(G_0)$
3. What else?
   - Still open problems

Of central importance to the theory of Borel equivalence relations is the *Vitali's equivalence relation* E_0:

$$xE_0y \leftrightarrow \forall^\infty n(x(n) = y(n)).$$

The Glimm-Effros Dichotomy says this is the *least non-smooth Borel equivalence relation*:

Theorem (Glimm-Effros Dichotomy, see [HKL90])

Let X be a Polish space and E be a Borel equivalence relation on X. Then one of the following holds:

(a) E is smooth — i.e., E is Borel reducible to equality on some Polish space Y —, or

(b) there is a continuous embedding from E_0 to E.

The graph counterpart of E_0 is the graph G_0 defined as follows:

Let $(s_n)_{n\in\omega}$ be a dense sequence of elements of $2^{<\omega}$ such that

- $|s_n| = n$ for all $n \in \omega$, and
- every $t \in 2^{<\omega}$ has an extension of the form s_n.

The G_0-graph is the graph on 2^ω defined by

$$G_0 \doteq \{(s_n^\frown 0^\frown x, s_n^\frown 1^\frown x) \mid n \in \omega \wedge x \in 2^\omega\}.$$

Let X be a Polish space and G be a graph on it.

The Borel chromatic number of G, denoted by $\chi_B(G)$, is the least cardinality of a family of Borel G-indendent[1] sets covering X.

It turns out $\chi_B(G_0) \geq \aleph_0$. In fact,

$$\chi_B(G_0) \geq \mathrm{cov}(\mathcal{M}). \tag{1}$$

This follows from the fact that any Borel (Baire measurable) G_0-independent set has to be meager (see Proposition 6.2. from [KST99]).

Also, this is the *minimal* analytic graph with uncountable Borel chromatic number, in the following sense:

[1] $B \subseteq X$ is G-independent iff $B^2 \cap G = \emptyset$.

Theorem (G_0-dichotomy, [KST99])

Let X be a Polish space and G be an analytic graph on X, then exactly one of the following holds:

(a) either $\chi_B(G) \leq \aleph_0$, or
(b) there is a continuous homomorphism from G_0 to G. In which case $\chi_B(G_0) \leq \chi_B(G)$.

The importance of the above dichotomy is highlighted by Ben Miller [Mil12] who showed how this implies many well-known descriptive set-theoretic dichotomy-theorems.

In contrast to the the case with the meager sets, there is Borel G_0-independent set of *positive Lebesgue measure*.

In fact, it is is open whether $\chi_B(G_0) \geq \mathrm{cov}(\mathcal{N})$ can be proved in ZFC[2]

On the other hand, for the bigger relative of G_0, the graph G_1, we have such inequality:

Let G_1 be the graph on 2^ω defined as

$$G_1 = \{(x, y) \mid \exists! n(x(n) \neq y(n))\}.$$

[2] We conjecture that these cardinals are orthogonal to each other.

Using the Lebesgue density theorem one can argue that any Borel (Lebesgue measurable) G_1-independent subset of 2^ω has measure zero.

It follows that

$$\chi_B(G_1) \geq \mathrm{cov}(\mathcal{N}). \tag{2}$$

Finally, we define $\chi_B(E_0)$ as the least cardinality of a family of Borel partial transversals covering 2^ω.

Since $G_0 \subseteq G_1 \subseteq E_0$, we trivially have

$$\chi_B(G_0) \leq \chi_B(G_1) \leq \chi_B(E_0). \tag{3}$$

The last ZFC-inequality connects $\chi_B(G_1)$ with the reaping number $\mathfrak{r}$:

$$\chi_B(G_1) \leq \mathfrak{r}. \tag{4}$$

We will see this is connected to the fact that Silver forcing adds splitting reals.

The proof is exactly the same as in Brendle's [Bre94] proof of $\mathrm{cov}(v^0) \leq \mathfrak{r}$, where v^0 is the σ-ideal os Silver null sets.

Combining these four inequalities with other known inequalities between cardinals from the Cichon's and the van Douwen's diagrams, we get the following picture:

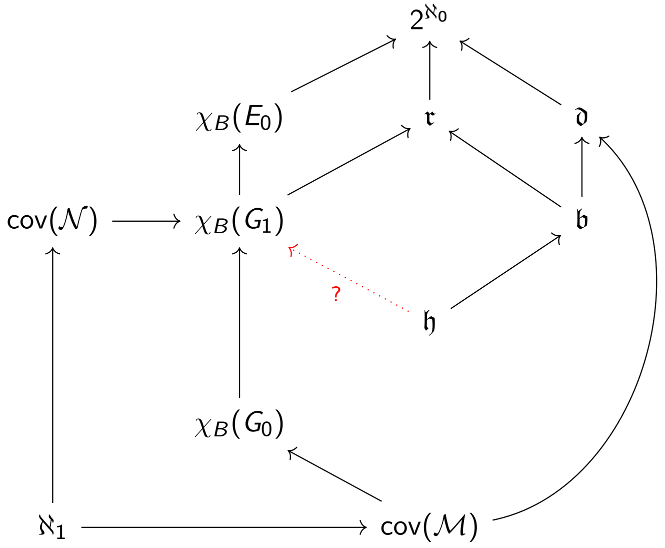

Figure: G_0-diagram.

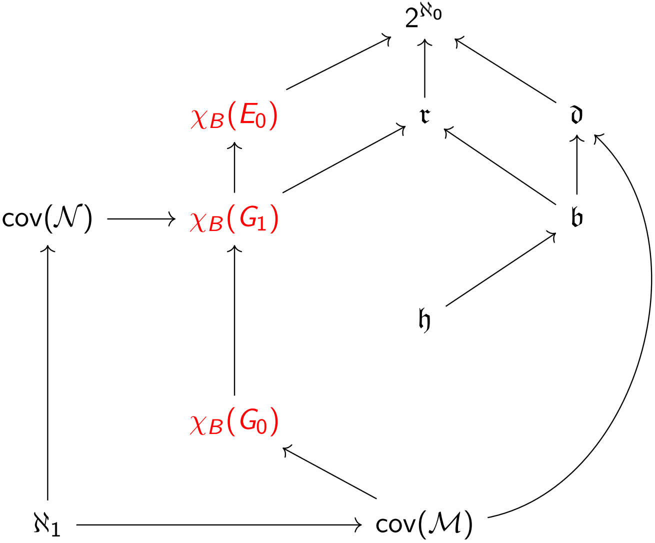

Figure: G_0-diagram.

For G a graph on a Polish space X, we let I_G denote the σ-ideal generated by Borel G-independent sets.

The natural forcing notion candidate to increase $\chi_B(G)$ is $\mathrm{Bor}(2^\omega) \setminus I_G$.

Of course, it needs to be checked for which graphs this forcing notion is actually proper. For this purpose, it is often useful to find a forcing notion with perfect trees that densely embeds into $\mathrm{Bor}(2^\omega) \setminus I_G$.

Lemma (Miller Paris' notes)

Let X be a Polish space and G be an analytic graph on it. If A is an analytic G-independent set, then there is some Borel set $B \supseteq A$ such that B is G-independent.

For an analytic graph G on 2^ω, say that a perfect tree p on $2^{<\omega}$ is a G-tree iff it is perfect, and

$$\forall s \in p\ ([p_s] \text{ has a } G\text{-edge}).$$

Let $\mathbb{P}_G$ denote the consisting of G-trees ordered by inclusion.

At a first glance, we could think the ideal I_G always has the *inner approximation property* — i.e., the compact I_G-positive sets are dense in the set of Borel I_G-positive sets.

We don't know a general proof of this fact, but this holds for the situations in which we have actual embeddings of G_0:

Theorem ([KST99])

Let X be a Polish space and G be an analytic graph on X which is either acyclic or locally countable, then exactly one of the following holds:

(a) either $\chi_B(G) \leq \aleph_0$, or

(b) there is a continuous **embedding** from G_0 to G.

Theorem

Let G be an analytic graph on 2^ω which is either acyclic or locally countable and A be an analytic set. Then either $A \in I_G$ or there is some G-tree p such that $[p] \subseteq A$.

Consider the graph defined as $G \restriction A = G \cap (A \times A)$. Using the G_0-dichotomy, one of the following holds:

(a) $\chi_B(G \restriction A) \leq \aleph_0$.

In this case, let $(B_n)_{n\in\omega}$ be a sequence of Borel $(G \restriction A)$-independent sets such that

$$2^\omega = \bigcup_{n\in\omega} B_n.$$

Now each $B_n \cap A$ is an analytic G-independent set. For each n we find a Borel $C_n \supseteq B_n \cap A$ which is G-independent.

(b) there is continuous embedding from G_0 to G_A.

In this case, let $\varphi : 2^\omega \to 2^\omega$ be the continuous embedding from G_0 to $G \restriction A$.

Note that $\varphi[2^\omega] \subseteq A$ is a compact I_G-positive set (here we need the injectivity of φ).

Moreover, $\varphi[O]$ has a G-edge for every open set O (since O has a G_0-edge).

From this, it is possible to show that $\varphi[2^\omega]$ is the set of branches of a G-tree.

Let G, H be two analytic graphs on the Cantor space that are either acyclic or locally countable, both having uncountable Borel chromatic number.

Assume we can prove $\chi_B(H) \leq \chi_B(G)$ also and the forcing $\mathbb{P}_G$ is a proper forcing notion.

The standard forcing recipe to prove the consistency of $\chi_B(H) < \chi_B(G)$ is:

(1) Prove that forcing with $\mathbb{P}_G$ does not add $\mathbb{P}_H$-quasi-generic reals.

(2) Assume CH in the ground model and let $(\mathbb{P}_G)_{\omega_2}$ be a countable supported iteration of $\aleph_2$ copies of $\mathbb{P}_H$ and prove that no $\mathbb{P}_H$-reals appear in successor steps of this iteration.

(3) Prove the same as above for limit steps.

For a natural number n, we define an *action* of 2^n on 2^ω as follows: For $\sigma \in 2^n$ and $x \in 2^\omega$, then $\sigma \cdot x$ replaces $x \upharpoonright n$ with σ in x — i.e.,

$$(\sigma \cdot x)(m) = \begin{cases} \sigma(m) \text{ if } m < n \\ x(m) \text{ if } m \geq n \end{cases}$$

- An $\mathbb{E}_0$-tree p is a perfect tree such that for any splitting node $\sigma \in p$, there are τ_0, τ_1, extensions of the same lenght, such that
$$\tau_1 \cdot [p_{\tau_0}] = [p_{\tau_1}].$$
- If τ_0, τ_1 can always be chosen in a way that $|\tau_0 \Delta \tau_1| = 1$, then p is a Silver tree.

Clearly, any $\mathbb{E}_0$-tree is an E_0-tree; and any Silver tree is a G_1-tree. In fact, these forcing notions are respectively equivalent — see Zapletal [Zap04].

The $\mathbb{E}_0$-forcing and the Silver forcing $\mathbb{V}$ are relatively well-known forcing notions and our natural candidates to increase *only* $\chi_B(E_0)$ and $\chi_B(G_1)$, respectively, while keeping the other cardinals of our diagram intact.

Lemma

(a) $\mathbb{E}_0$ does not add Silver-quasi-generic reals.

(b) $\mathbb{V}$ does not add $\mathbb{P}_{G_0}$-reals.

For any forcing notion $\mathbb{P}$ and $\dot{x}$ a $\mathbb{P}$-name for an element of 2^ω witnessed by p, define for each $q \leq p$,

$$T_q(\dot{x}) = \{s \in 2^{<\omega} \mid \exists r \leq q(r \Vdash s \subseteq \dot{x})\},$$

the tree of q-interpretations for $\dot{x}$. We have

$$q \Vdash \dot{x} \in [T_q(\dot{x})]$$

and each $[T_q(\dot{x})]$ is a closed set coded in the ground model.

In case the case of (a) the goal is to find $q \leq p$ an $\mathbb{E}_0$-tree such that $[T_q(\dot{x})]$ is a G_1-independent set.

Likewise, in the case of (b) the goal is to find $q \leq p$ a Silver tree such that $[T_q(\dot{x})]$ is a G_0-independent set.

Theorem

The countable supported iteration of $\aleph_2$ copies of the $\mathbb{E}_0$-forcing, over a model of CH, yields to a model of

$$\aleph_1 = \chi_B(G_1) < \chi_B(E_0) = 2^{\aleph_0}.$$

Theorem

The countable supported iteration of $\aleph_2$ copies of the Silver forcing, over a model of CH, yields to a model of

$$\aleph_1 = \chi_B(G_0) < \chi_B(G_1) = 2^{\aleph_0}.$$

We would like to separate $\mathrm{cov}(\mathcal{M})$ from $\chi_B(G_0)$. It is tempting to say that, just like Silver and $\mathbb{E}_0$-forcing, $\mathbb{P}_{G_0}$ is proper and has the Sacks property.

Surprisingly, this is far from the truth: Zapletal [Zap08] (Theorem 4.7.20) proved that forcing with G_0-trees is not proper. In fact, it collapses the continuum to $\aleph_0$.

This problem arises from the fact that G_0 is not a very homogeneous graph — i.e.,

If B is a Borel I_{G_0}-positive subset of $[s_n]$, there is a compact I_{G_0}-positive set $C \subseteq B$ such that the *bit-flipped set*

$$\pi_n[C] \doteq \{s_n^\frown i^\frown x \in 2^\omega \mid s_n^\frown (1-i)^\frown x \in C \text{ for } i < 2\}$$

is a G_0-independent set (Claim 4.7.21 of [Zap08] due to Ben Miller).

From this, Zapletal shows that this implies that any B Borel I_{G_0}-positive set is compatible with uncountable many elements of a maximal antichain inside some $[s_n]$:

$$A_n \subseteq \{C \subseteq [s_n] | \ \pi_n[C] \textit{ is } I_{G_0}\textit{-small}\}.$$

We fix this problem by eliminating such sets from our conditions.

This yields to some forcing between Cohen and $\mathbb{P}_{G_0}$.

Say that p is a *fat G_0-tree* iff

- it is a Silver tree, and
- for every splitting node $\tau \in p$ and for every $\sigma \in 2^{|\tau|}$

$$\tau \cdot [p_\sigma] \text{ has a } G_0\text{-edge.}$$

Denote the fat G_0-forcing by $\mathbb{P}^f_{G_0}$.

Theorem

The fat G_0-forcing is an ω^ω-bounding proper forcing notion. Therefore, it does not add Cohen reals.

The ω^ω-bounding is preserved for countable supported iterations of proper forcing notions.

As a consequence, the countable supported iteration of $\aleph_2$ copies of $\mathbb{P}^f_{G_0}$ yields to a model of $\mathrm{cov}(\mathcal{M}) < \chi_B(G_0)$.

Say that $q \leq_0^f p$ iff q and p have the same stem.

Let τ be the stem of q and for $\sigma \in 2^{|\tau|}$ and n_σ be the least natural n number such that s_n is a splitting node of $\sigma \cdot [p_\tau]$.

Say that $\sigma' \in L_1^f(q)$ iff it is a splitting node of q and any proper initial segment that is a splitting node of q has height among the levels n_σ's.

Say that $q \leq_1^f p$ iff $L_1^f(q) = L_1^f(p)$.

Assume we have defined $L^f_n(p)$ and repeat the same procedure as before to define $L^f_{n+1}(q)$:

For $\tau \in L^f_n(q)$, let $\sigma \in 2^{|\tau|}$ and choose n_σ the least natural number for which s_{n_σ} is a splitting node of $\sigma \cdot [p_\tau]$.

The elements of $L^f_{n+1}(q)$ will then be splitting nodes of q such that any proper initial segment that is a splitting node of q has height among the levels n_σ's, for $\tau \in L^f_n(q)$ and $\sigma \in 2^{|\tau|}$.

Now that $L^f_n(q)$ is defined for every q, we say that $q \leq^f_n p$ iff $q \leq p$ and $L^f_n(q) = L^f_n(p)$.

Check that if $(q_n)_{n\in\omega}$ is a sequence such that $q_{n+1} \leq_{n+1} q_n$, then $q = \bigcap q_n$ is still a fat G_0-tree.

Let A be an antichain on $\mathbb{P}^f_{G_0}$, p be any condition and $n \in \omega$. We aim there is $q \leq^f_n p$ compatible with at most finitely many elements of A[3]:

Enumerate $L^f_n(p) = \{\tau_0, ..., \tau_m\}$ and let $r_0 \leq p_{\tau_0}$ be a condition compatible with at most one element of A. Define r_i to be the *amalgamation of* r_0 *into* p_{τ_i}:

$$[r_i] = \tau_i \cdot [r_0].$$

Finally, let q_0 be the union of all r_i. Repeating this process with q_0 now we obtain a sequence $q_1, ..., q_m$ and we let $q = q_m$.

By construction we have $q \leq^f_n p$ and it is compatible with at most m elements of the antichain A.

[3] This is the strong form of the Axiom A, which implies ω^ω-boundedness as well

What have we lost: the idea with closing up G_0-edges for the actions of binary trees was imported from the Silver forcing.

A closer inspection on the Silver forcing will show that the same amalgamation technique can be used to prove that it adds *reals of minimal degree* and that it has the Sacks property.

Unfortunately it is not clear that the fat G_0-forcing also has the Laver property, which would imply that no random reals were added and, moreover, $|\mathrm{cof}(\mathcal{N})|$ has the value of the ground-model continuum[4].

[4] Stefan thinks it has the Sacks property, but minimality is more mysterious

- **The relationship between the graph and its forcing.**

  For an analytic graph G on the Cantor space, acyclic or locally countable, with $\chi_B(G) > \aleph_0$, we see that the correspondent forcing $\mathbb{P}_G$ may or may not be proper.

  It is natural to investigate how properties of the graph affect properties of the respective forcing.

  One idea: note that $\mathbb{V}$ is equivalent to its fat version, while this is not true for the G_0-forcing:

  Any Silver tree p has the property: for every splitting node $\tau \in p$ and every $\sigma \in 2^{|\tau|}$,

$$\sigma \cdot [p_\tau] \text{ has a } G_1\text{-edge.}$$

- **ZFC-proof of $\mathfrak{h} \leq \chi_B(G_1)$.**

  Since
  $$\mathfrak{h} \leq \mathrm{cov}(r^0 \cap \mathrm{Bor}([\omega]^\omega)),$$
  We could try to show that G_1-independent Borel sets can be coded into Ramsey-null sets.

  Yurii Khomskii observed this trivially follows from the Ramsey property for Borel sets.

- **Consistency of $\chi_B(G_0) < \mathrm{cov}(\mathcal{N})$.**

  Note that the inequality $\mathrm{cov}(\mathcal{N}) < \chi_B(G_0)$ holds in the Cohen model.

  One can check that any random real (over the ground model) is always in some closed G_0-independent positive-measure set coded in the ground model. What about any other real?

- **Consistency of $\chi_B(E_0) < \mathfrak{d}$.**

  It is known that E_0-forcing has the Sacks property (therefore it is ω^ω-bounding). This yields to the consitency of $\mathfrak{d} < \chi_B(E_0)$. It is clear that in the Miller model $\chi_B(G_1) < \mathfrak{d}$, but this is open for $\chi_B(E_0)$.

- **Consistency of $\chi_B(G_1) < \mathfrak{b}$.**

  This is related to the old question "Does Laver forcing add Silver-quasi-generic reals?"

  This obviously relates to the implications and non-implications between Silver and Laver measurability.

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Thank you!