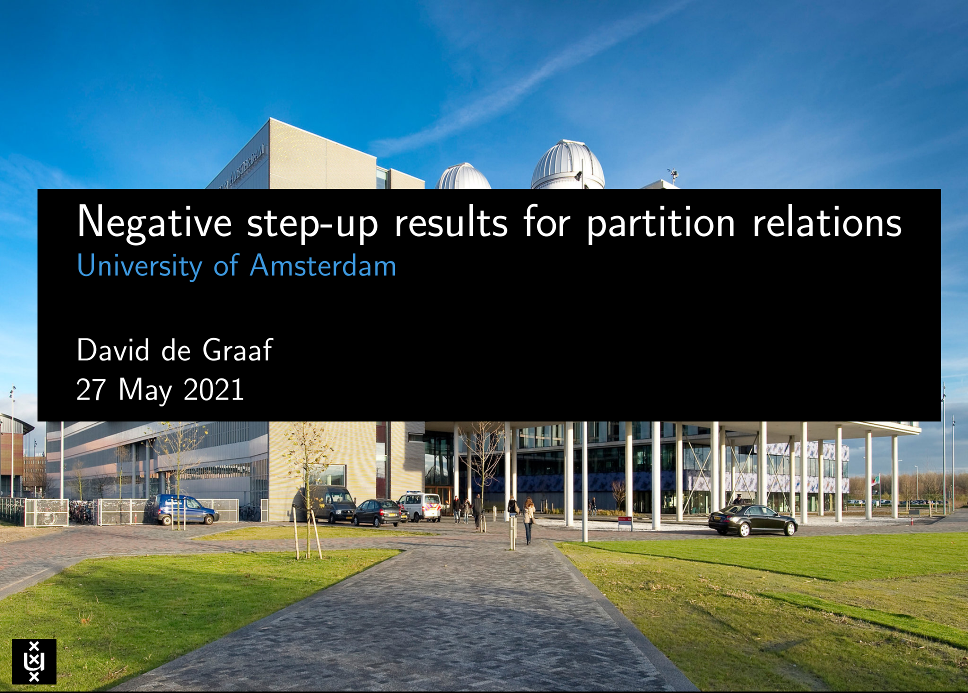

Negative step-up results for partition relations

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Colourings

- $[S]^r = \{A \subseteq S \colon |A| = r\}$.
- A *k-colouring* of $[S]^r$ is a function $f\colon [S]^r \to k$.
- Given $f\colon [S]^r \to k$, a set $H \subseteq S$ is *i-homogeneous for f* if $f \restriction [H]^r$ is constant with colour $i \in k$.

Definition (Arrow notation)

Let α and β_i be order-types for all $i < m$, where m is a cardinal and let $r \in \mathbb{N}$. We write

$$\alpha \to (\beta_i)^r_{i<m},$$

if for all sets S with $\operatorname{otp} S = \alpha$ and every m-colouring $f\colon [S]^r \to m$ there exists a i-homogeneous set $H \subseteq S$ with $\operatorname{otp} H = \beta_i$.

- α is the *resource*,
- β_i are the *goals*,
- r is the *exponent*, and
- m is the *colour set* or *colour cardinal*.

Positive Step-Up Lemma

Theorem (Theorem 39, [1])

For all infinite cardinals κ, finite r, any cardinal m and any ordinal λ, if $\kappa \to (\lambda)^r_m$, then $(2^{<\kappa})^+ \to (\lambda+1)^{r+1}_m$.

Theorem

For any $r, k \in \mathbb{N}$, $\omega_1 \to (\omega+1)^r_k$.

Theorem

Let $r, k \in \mathbb{N}$. Then $(2^{\aleph_0})^+ \to (\omega+2)^r_k$.

Theorem (Erdős-Rado Theorem)

For any infinite cardinal κ, $\beth_n(\kappa)^+ \to (\kappa^+)^{n+1}_\kappa$.

Pattern of partition relations

Theorem

For any $r, k \in \mathbb{N}$, $\omega_1 \to (\omega+1)^r_k$.

Theorem

Let $r, k \in \mathbb{N}$. *Then* $(2^{\aleph_0})^+ \to (\omega+2)^r_k$.

- $\beth_0^+ \to (\omega+1)^r_k$.
- $\beth_1^+ \to (\omega+2)^r_k$.
- $\beth_2^+ \to (\omega+3)^r_k$.
- etc...

Pattern of partition relations

- $\beth_0^+ \to (\omega+1)^r_k$.
- $\beth_0^+ \not\to (\omega+2)^3_2$.
- $\beth_1^+ \to (\omega+2)^r_k$.
- $\beth_1^+ \not\to (\omega+3)^4_2$.
- $\beth_2^+ \to (\omega+3)^r_k$.
- $\beth_2^+ \not\to (\omega+4)^5_2$?

Theorem (Lemma 4, [1])

Let $\alpha_0, \alpha_1, \beta_0, \beta_1$ *be linear order-types and* $2 \leq r < \omega$. *Assume that* $|\alpha_0| = |\alpha_1|$ *and* $\beta_0, \beta_1^* \not\leq \alpha_0$. *Then*

$$\alpha_1 \not\rightarrow (\beta_0, \beta_1, (r+1)_{r!-2})^r_{r!}.$$

Proof.

Throughout we may assume $|\beta_0|, |\beta_1| \geq \aleph_0$. Let S be a set such that $\mathrm{otp}(S, <) = \alpha_1$. As $|S| = |\alpha_1| = |\alpha_0|$, there is an ordering $\ll$ on S such that $\mathrm{otp}(S, \ll) = \alpha_0$. Given any $X \in [S]^r$, we can index the elements in X such that $X = \{x_0 < x_1 < \ldots < x_{r-1}\}$. There is a unique permutation $\pi \colon r \to r$ such that $x_{\pi(0)} \ll x_{\pi(1)} \ll \ldots \ll x_{\pi(r-1)}$. Note that there are precisely $r!$ permutations of r. Fix an enumeration $\langle \pi_n \mid n < r! \rangle$ of permutations of r, where π_0 is the identity, and $\pi_1 = \pi_0^*$.

Proof (continued).

Define the $r!$-colouring $f\colon [S]^r \to r!\colon \{x_0 < x_1 < \ldots < x_{r-1}\} \mapsto n$ where π_n is such that $x_{\pi_n(0)} \ll x_{\pi_n(1)} \ll \ldots \ll x_{\pi_n(r-1)}$. Suppose there is a n-homogeneous set H for f with $\mathrm{otp}(H,<) = \beta_n$, there are three cases that we consider.

Case $n = 0$. Then $\mathrm{otp}(H,<) = \beta_0$ and $f \restriction [H]^r \equiv 0$. As π_0 is the identity and $r \geq 2$, we have in particular for any $x, y \in H$ that $x < y \iff x \ll y$. This means $\beta_0 = \mathrm{otp}(H,<) = \mathrm{otp}(H,\ll) \leq \mathrm{otp}(S,\ll) = \alpha_0$, which is a contradiction.

Case $n = 1$. Then $\mathrm{otp}(H,<) = \beta_1$ and $f \restriction [H]^r \equiv 1$. In this case $\pi_1 = \pi_0^*$, which means for any $x, y \in H$ we have $x < y \iff y \ll x$. Therefore $\beta_1^* = \mathrm{otp}(H,<)^* = \mathrm{otp}(H,\ll) \leq \mathrm{otp}(S,\ll) = \alpha_0$, again a contradiction.

Proof (continued).

Case $n \geq 2$. Then $\mathrm{otp}(H,<) = r+1$ and $f \restriction [H]^r \equiv n$. In particular $\pi_n \neq \pi_0$ and $\pi_n \neq \pi_1$. Write $H = \{x_0 < x_1 < \ldots < x_{r-1} < x_r\}$ and define $y_k = x_{k+1}$. Then

$$x_{\pi_n(0)} \ll x_{\pi_n(1)} \ll \ldots \ll x_{\pi_n(r-1)},$$
$$y_{\pi_n(0)} \ll y_{\pi_n(1)} \ll \ldots \ll y_{\pi_n(r-1)}.$$

Suppose $x_0 \ll x_1$, then $x_{\pi_n(0)^{-1}} < x_{\pi_n(1)^{-1}}$ and so $y_0 \ll y_1$. This gives $x_1 \ll x_2$. Repeating this argument gives that $x_0 \ll x_1 \ll \cdots \ll x_{r-1}$, and hence $\pi_n = \pi_0$, which is a contradiction. If, on the other hand, we assume $x_1 \ll x_0$, then using an analogous argument, we get $x_{r-1} \ll \ldots \ll x_1 \ll x_0$, i.e. $\pi_n = \pi_1$, which is also a contradiction. We conclude that such a homogeneous set H cannot exist, and this concludes the proof. □

Theorem (Lemma 5, [1])

Let α be an ordinal and $r < \omega$ and m any cardinal. Let γ_n be ordinals for all $n < m$ such that $\beta \not\to (\gamma_n)^r_{n<m}$ for all $\beta < \alpha$. Then $\alpha \not\to (\gamma_n + 1)^{r+1}_{n<m}$.

Proof.

Let S be a set such that $\mathrm{otp}(S, <) = \alpha$. For every $x \in S$, define $I_x = \{y \in S \mid y < x\}$, then $\mathrm{otp}(I_x, <) = \beta < \alpha$, for some β. By assumption there is some colouring $f_x \colon [I_x]^r \to k$ such that there is no n-homogeneous set H of order-type γ_n.

Define $f \colon [S]^{r+1} \to k$ by

$$\{x_0 < x_1 \ldots < x_{r-1} < x_r\} \mapsto f_{x_r}(\{x_0 < x_1 \ldots < x_{r-1}\}).$$

If there is an n-homogeneous set $H = \{h_i \mid i < \gamma_n + 1\} \subseteq S$ for f with $\mathrm{otp}(H, <) = \gamma_n + 1$, then the set $\{h_i \mid i < \gamma_n\}$ is n-homogeneous for $f_{h_{\gamma_n}}$ with order-type γ_n. This is a contradiction and hence the proof is concluded. □

The first negative relation

Theorem (Theorem 41, [1])

$\omega_1 \not\to (\omega + 2)^3_2$.

Proof.

Clearly, $\omega + 1, \omega^* \not\leq \omega$. By Lemma 4, we have $\beta \not\to (\omega + 1, \omega)^2_2$ for all countable ordinals β. This relation holds for all $\beta < \omega_1$ and thus by Lemma 5, $\omega_1 \not\to (\omega + 2, \omega + 1)^3_2$. □

Recapitulate

- $\beth_0^+ \to (\omega+1)^r_k$.
- $\beth_0^+ \not\to (\omega+2)^3_2$.
- $\beth_1^+ \to (\omega+2)^r_k$.
-
- $\beth_2^+ \to (\omega+3)^r_k$.

$$\beth_1^+ \not\to (\omega+3)^4_2.$$

Discrepancy

Definition

Given distinct $f, g \in 2^\kappa$, we define the *discrepancy* δ as

$$\delta(f, g) = \min\{\xi < \kappa \mid f(\xi) \neq g(\xi)\}.$$

If $f = g$, we simply let $\delta(f, g) = \kappa$.

Observation

If $\delta(f, g) < \delta(g, h)$, then $\delta(f, g) = \delta(f, h)$.

Remark

Let $\prec$ denote the lexicographic ordering on 2^κ. If $f, g, h \in 2^\kappa$ are such that $f \prec g \prec h$, then $\delta(f, g) \neq \delta(g, h)$. Else, $f(\xi) < g(\xi) < h(\xi)$ for some $\xi < \kappa$.

Result by Albin Jones

Theorem (Albin L. Jones (2000), [3])

Let α be a linear order-type and let κ be an infinite cardinal. If $\alpha \not\to (\omega)^1_{2^\kappa}$, then $\alpha \not\to (\kappa+2,\omega)^3_2$.

Proof.

Let $e\colon \alpha \to 2^\kappa$ be a witness of $\alpha \not\to (\omega)^1_{2^\kappa}$. As ω is regular, it follows for every $B \in [\alpha]^\omega$ there is $C \in [B]^\omega$ such that $e \restriction C$ is injective. Define the partition $f\colon [\alpha]^2 \to \kappa+1$ by $\{x,y\} \mapsto \delta(e(x),e(y))$.

Define the partition of triples $g\colon [\alpha]^3 \to 2$ s.t. for $x<y<z$,

$$g\{x,y,z\} = \begin{cases} 0 & \text{if } e \text{ is injective on } \{x,y,z\} \text{ and } f\{x,y\} < f\{y,z\}, \text{ and} \\ 1 & \text{if } e \text{ is not injective on } \{x,y,z\} \text{ or } f\{x,y\} \geq f\{y,z\}. \end{cases}$$

We show that g is the partition which proves $\alpha \not\to (\kappa+2,\omega)^3_2$.

Claim

There is no 0*-homogeneous* $H \subseteq \alpha$ *for* g *with* $\operatorname{otp} H = \kappa + 2$.

Proof of claim.

Suppose such $H = \{h_\gamma \mid \gamma < \kappa + 2\}$ exists. We observe immediately that $e \restriction H$ is injective. In particular, $e(h_\kappa) \neq e(h_{\kappa+1})$ and hence $f\{h_\kappa, h_{\kappa+1}\} = \delta(e(h_\kappa), e(h_{\kappa+1})) = \xi < \kappa$. For any $\mu < \nu < \kappa$ we have $f\{h_\mu, h_\nu\} < f\{h_\nu, h_\kappa\}$, and by the observation:

$$f\{h_\mu, h_\kappa\} = f\{h_\mu, h_\nu\} < f\{h_\nu, h_\kappa\}.$$

Note that $f\{h_\mu, h_\kappa\} < f\{h_\kappa, h_{\kappa+1}\} = \xi < \kappa$. Hence, the sequence $\langle f\{h_\mu, h_\kappa\} \mid \mu < \kappa\rangle$ is a strictly increasing sequence of length κ of ordinals below ξ, which gives a contradiction. ∎

Claim

There is no 1*-homogeneous* $H \subseteq \alpha$ *for* g *with* $\operatorname{otp} H = \omega$*.*

Proof of claim.

Again, for sake of contradiction assume such $H \in [\alpha]^\omega$ exists. By the remark above there is $B \in [H]^\omega$ such that $e \restriction B$ is injective. Consider the colouring $h\colon [B]^3 \to 2$ by

$$h\{x < y < z\} = \begin{cases} 0 & \text{if } f\{x,y\} > f\{y,z\}, \text{ and} \\ 1 & \text{if } f\{x,y\} = f\{y,z\}. \end{cases}$$

By definition of g and since B is 1-homogeneous for g, the colouring h is well-defined. Now, by a weak version of Ramsey's Theorem, $\omega \to (\omega, 4)^3_2$. Hence, either

(a) there is $C \in [B]^\omega$ such that $h \restriction [C]^3 \equiv 0$, or

(b) there is $D \in [B]^4$ such that $h \restriction [D]^3 \equiv 1$.

If (a) holds, then $\langle f\{c_n, c_{n+1}\} \mid n \in \omega\rangle$, where $C = \{c_0 < c_1 < \dots\}$, is a strictly decreasing sequence of ordinals of length ω, which is a contradiction.

Proof of claim (continued).

Alternatively, if (b) holds and such $D = \{x < y < z < w\}$ exists, then $f\{x, y\} = f\{y, z\} = f\{x, z\}$. This gives us three pairwise distinct functions $e(x), e(y), e(z) \in 2^\kappa$ such that they are pairwise different at some point $\xi < \kappa$, which is not possible since these functions map to 2. Hence we reach a contradiction. ∎

The second negative relation

Proof (continued).

We conclude that

$$\alpha \not\to (\omega)^1_{2^\kappa} \implies \alpha \not\to (\kappa + 2, \omega)^3_2.$$

$\square$

Theorem

$\beth_1^+ \not\to (\omega + 3, \omega + 1)^4_2$.

Proof.

Let $\beta < \beth_1^+$, then $|\beta| \leq \beth_1 = 2^{\aleph_0}$. Define the partition $f \colon \beta \to \beta \colon \gamma \mapsto \gamma$. Then f witnesses $\beta \not\to (\omega)^1_\beta$. Therefore $\beta \not\to (\omega)^1_{2^{\aleph_0}}$.
By Jones's lemma, we have $\beta \not\to (\omega + 2, \omega)^3_2$.
Then by Lemma 4 in [1], we obtain the desired result. $\square$

Does the pattern continue?

- $\beth_0^+ \to (\omega+1)^r_k$.
- $\beth_0^+ \not\to (\omega+2)^3_2$.
- $\beth_1^+ \to (\omega+2)^r_k$.
- $\beth_1^+ \not\to (\omega+3)^4_2$.
- $\beth_2^+ \to (\omega+3)^r_k$.

- $\beth_2^+ \not\to (\omega+4)^5_2$?

Converse Positive Step-Up Lemma unprovable

The converse: $\kappa \not\to (\lambda)^r_k \implies (2^{<\kappa})^+ \not\to (\lambda+1)^r_k$.

Assume $2^{\aleph_0} = \aleph_2$ and $2^{\aleph_1} = 2^{\aleph_2} = \aleph_3$. Then $\beth_2^+ = (2^{2^{\aleph_0}})^+ = \aleph_4$ and $(2^{<\aleph_2})^+ = \aleph_4$.

- Sierpińsky: $\aleph_2 = 2^{\aleph_0} \not\to (\aleph_1)^2_2$
- Erdős-Rado: $\aleph_4 = \beth_2^+ \to (\aleph_1 + 1)^3_2$.

The idea

- We want to show $\kappa \not\to (\alpha)^r_m \implies 2^\kappa \not\to (\alpha+1)^{r+1}_m$.
- We will show $\kappa \not\to (\lambda)^r_m \implies 2^\kappa \not\to (\lambda)^{r+1}_m$.
- Given a partition $\langle I_\xi \mid \xi < m \rangle$ witnessing $\kappa \not\to (\lambda)^r_m$.
- Want to create partition $\langle I^*_\xi \mid \xi < m \rangle$ witnessing $2^\kappa \not\to (\lambda)^{r+1}_m$.
- The partition will (unfortunately) only work if λ is a cardinal.

A bunch of definitions

Let $r \geq 3$ and $u \in [2^\kappa]^r$. Write $u = \{x_0 <^* x_1 <^* \ldots <^* x_{r-1}\}$ and define

- $\eta(u) = (\eta(x_0, x_1), \eta(x_1, x_2), \ldots, \eta(x_{r-2}, x_{r-1}))$,

where $\eta(x, y) = 0$ if $x <^* y \iff x \prec y$, and $\eta(x, y) = 1$ otherwise.

Also given $s \leq r - 1$ and $k_0, k_1, \ldots, k_{s-1} \in 2$, define

- $K(k_0, k_1, \ldots, k_{s-1}) = \{u \in [2^\kappa]^r \mid \eta(u) \restriction s = (k_0, k_1, \ldots, k_{s-1})\}$.
- $K_0 = K(0, 0, \ldots, 0)$ and $K_1 = K(1, 1, \ldots, 1)$.
- $K = K_0 \cup K_1$.

Examples

Given $u = \{x_0 <^* x_1 <^* \ldots <^* x_{r-1}\}$.

$$\begin{aligned} u \in K(0,1) &\iff \eta(x_0, x_1) = 0 \text{ and } \eta(x_1, x_2) = 1 \\ &\iff x_0 \prec x_1 \succ x_2. \end{aligned}$$

- $u \in K_0 \iff x_0 \prec x_1 \prec \ldots \prec x_{r-1}$
- $u \in K_1 \iff x_0 \succ x_1 \succ \ldots \succ x_{r-1}$

More definitions

Let $r \geq 4$ and $u \in K$. Write $u = \{x_0 <^* x_1 <^* \ldots <^* x_{r-1}\}$ and define $\delta_s = \delta(x_s, x_{s+1})$ for $s \leq r-2$. Define

- $\zeta(\delta(u)) = (\zeta(\delta_0, \delta_1), \zeta(\delta_1, \delta_2), \ldots, \zeta(\delta_{r-3}, \delta_{r-2}))$,

where $\zeta(\delta_s, \delta_{s+1}) = 0$ if $\delta_s < \delta_{s+1}$, and $\zeta(\delta_s, \delta_{s+1}) = 1$ if $\delta_s > \delta_{s+1}$.

Also given $s \leq r-2$ and $k_0, k_1, \ldots, k_{s-1} \in 2$, define

- $P(k_0, k_1, \ldots, k_{s-1}) = \{u \in K \mid \zeta(\delta(u)) \restriction s = (k_0, k_1, \ldots, k_{s-1})\}$.
- $P_0 = P(0, 0, \ldots, 0)$ and $P_1 = P(1, 1, \ldots, 1)$.
- $P = P_0 \cup P_1$.

More examples

Given $u = \{x_0 <^* x_1 <^* \ldots <^* x_{r-1}\}$.

$$
\begin{aligned}
u \in P(0,1) &\iff \zeta(\delta_0, \delta_1) = 0 \text{ and } \zeta(\delta_1, \delta_2) = 1 \\
&\iff \delta_0 < \delta_1 > \delta_2 \\
&\iff \delta(x_0, x_1) < \delta(x_1, x_2) > \delta(x_2, x_3).
\end{aligned}
$$

- $u \in P_0 \iff \delta_0 < \delta_1 < \ldots < \delta_{r-2}$
- $u \in P_1 \iff \delta_0 > \delta_1 > \ldots > \delta_{r-2}$

Lemma (Lemma 23.12, [2])

Let $r \geq 3$, let κ be a cardinal. Let $I \subseteq [\kappa]^{r-1}$ and put

$$I^* = \{u \in P_0 \mid \delta(u) \in I\}. \tag{1}$$

Assume that $[H]^r \subseteq I^*$ for some $H \neq \emptyset$ where by assumption $\mathrm{otp}(H, <^*) = \alpha$. Then there is $X \subseteq \kappa$ with $\mathrm{otp}(X, <) = \alpha^-$ such that $[X]^{r-1} \subseteq I$.

Proof.

We may assume that $|H| \geq r$ and write $H = \{h_\gamma \mid \gamma < \alpha\}$ where $\alpha = \mathrm{otp}(H, <^*)$. (Recall that $<^*$ is a fixed well-order on 2^κ). For ordinals γ such that $\gamma + 1 < \alpha$ we let

$$\delta_\gamma = \delta(h_\gamma, h_{\gamma+1}).$$

Define

$$X = \{\delta_\gamma \mid \gamma + 1 < \alpha\}.$$

Proof (continued).

First we show that $\mathrm{otp}(X, <) = \alpha^-$. It obviously suffices to show for all $\gamma < \gamma' < \alpha^-$ that $\delta_\gamma < \delta_{\gamma'}$. By the assumption $[H]^r \subseteq I^* \subseteq P_0$, it follows that

$$\zeta(\delta(\{h_\gamma, h_{\gamma+1}, h_{\gamma'}\})) = \zeta(\delta(h_\gamma, h_{\gamma+1}), \delta(h_{\gamma+1}, h_{\gamma'})) = 0.$$

Also

$$\zeta(\delta(\{h_{\gamma+1}, h_{\gamma'}, h_{\gamma'+1}\})) = \zeta(\delta(h_{\gamma+1}, h_{\gamma'}), \delta(h_{\gamma'}, h_{\gamma'+1})) = 0.$$

In other words, $\delta_\gamma < \delta(h_{\gamma+1}, h_{\gamma'}) < \delta_{\gamma'}$. Note that we assumed $\gamma + 1 < \gamma'$, because if $\gamma + 1 = \gamma'$, we could just leave out the term $\delta(h_{\gamma+1}, h_{\gamma'})$. In particular, we obtain $\delta_\gamma < \delta_{\gamma'}$, showing that $\mathrm{otp}(X, <) = \alpha^-$.

Proof (continued).

It rests to show that $[X]^{r-1} \subseteq I$. Given $\xi_0 < \ldots < \xi_{r-2} < \alpha^-$, we want to show $\{\delta_{\xi_0} < \ldots < \delta_{\xi_{r-2}}\} \in I$. Suppose that $\xi_i + 1 < \xi_{i+1}$. As $[H]^r \subseteq P_0$, we have $\delta(h_{\xi_i}, h_{\xi_i+1}) < \delta(h_{\xi_i+1}, h_{\xi_{i+1}})$ and hence , $\delta(h_{\xi_i}, h_{\xi_i+1}) = \delta(h_{\xi_i}, h_{\xi_{i+1}})$. If $\xi_i + 1 = \xi_{i+1}$, then $\delta(h_{\xi_i}, h_{\xi_i+1}) = \delta(h_{\xi_i}, h_{\xi_{i+1}})$ obviously holds as well. Now, writing $\xi_{r-1} = \xi_{r-2} + 1$, we obtain

$$\begin{aligned}
\{\delta_{\xi_i} \mid i < r-1\} &= \{\delta(h_{\xi_i}, h_{\xi_i+1}) \mid i < r-1\} \\
&= \{\delta(h_{\xi_i}, h_{\xi_{i+1}}) \mid i < r-1\} \\
&= \delta(\{h_{\xi_i} \mid i < r\}).
\end{aligned}$$

As $\{h_{\xi_i} \mid i < r\} \in [H]^r \subseteq I^*$, we have by definition of I^* that $\{\delta_{\xi_i} \mid i < r-1\} \in I$. This gives us $[X]^{r-1} \subseteq I$, which is what we wanted to show. $\square$

Lemma (Lemma 23.5, [2])

Let $X \subseteq 2^\kappa$ and assume $|X| \geq \aleph_0$. Assume that (i) $[X]^r \cap K(0,1) = \varnothing$ or (ii) $[X]^r \cap K(1,0) = \varnothing$. Then there is a set $Y \subseteq X$ with $|Y| = |X|$ such that $[Y]^r \subseteq K_0$ or $[Y]^r \subseteq K_1$.

Proof.

Write $\lambda = |X|$ and we may assume $\mathrm{otp}(X, <^*) = \lambda$. Assume that no such Y exists.

Claim

There are elements $x_0 <^* x_1 <^* x_2 <^* x_3$ such that $x_0 \prec x_1 \succ x_2 \prec x_3$.

If the claim is proven, then there is $\{x_0, x_1, x_2, \dots\} \in [X]^r \cap K(0,1)$ and $\{x_1, x_2, x_3, \dots\} \in [X]^r \cap K(1,0)$, contradicting (i) or (ii), respectively, which gives the contradiction. Hence such Y exists. □

Claim

There are elements $x_0 <^* x_1 <^* x_2 <^* x_3$ *such that* $x_0 \prec x_1 \succ x_2 \prec x_3$.

Proof of claim.

For every $x \in X$ there are $y, z \in X$ and $y', z' \in X$ such that

$$x \leq^* y <^* z \text{ and } y \prec z, \tag{2}$$

$$x \leq^* y' <^* z' \text{ and } y' \succ z'. \tag{3}$$

Suppose not and let $x \in X$ be a counterexample, the set $Y = \{x' \in X \mid x \leq^* x'\}$ has cardinality λ and is contained in either K_0 or K_1, which is a contradiction.
Now let $x_0, z_1 \in X$ with $x_0 <^* z_1$ and $x_0 \prec z_1$. Then let $y_1, z_2 \in X$ with $z_1 \leq^* y_1 <^* z_2$ with $y_1 \succ z_2$. Define $x_1 = \max_{\prec}\{y_1, z_1\}$, then $x_0 <^* x_1$ and $x_0 \prec x_1$. Also, $x_1 \succ z_2$.
Pick $y_2, z_3 \in X$ with $z_2 \leq^* y_2 <^* x_3$ with $y_2 \prec x_3$. Let $x_2 = \min_{\prec}\{y_2, z_2\}$. Then $x_1 <^* x_2$ and $x_1 \succ x_2$. Also, $x_2 \prec x_3$. This proves the claim ∎

Lemma (Lemma 23.9, [2])

Let $r \geq 4$, let $X \subseteq 2^\kappa$ such that $|X| \geq \aleph_0$. Suppose $[X]^r \subseteq K_0$ or $[X]^r \subseteq K_1$. Assume (i) $[X]^r \cap P(0,1) = \varnothing$ or (ii) $[X]^r \cap P(1,0) = \varnothing$. Then there exists $Y \subseteq X$ with $|Y| = |X|$ such that $[Y]^r \subseteq P_0$.

Claim

Suppose $x_0 <^* x_1 <^* \ldots <^* x_{s-1}$ are such that

$$\zeta(\delta_i, \delta_{i+1}) \neq \zeta(\delta_{i+1}, \delta_{i+2}), \tag{4}$$

for all $i \leq s-4$. Then $s \leq 4$.

Proof of claim.

Suppose $s \geq 5$ and $x_0 <^* x_1 <^* x_2 <^* x_3 <^* x_4$ constitutes a counterexample. If $\zeta(\delta_0, \delta_1) < \zeta(\delta_1, \delta_2) > \zeta(\delta_2, \delta_3)$, then $\{x_0, x_1, x_2, x_3, \ldots\} \in [X]^r \cap P(0,1)$ or $\{x_1, x_2, x_3, x_4, \ldots\} \in [X]^r \cap P(1,0)$, giving a contradiction with (i) or (ii), respectively.

Similarly, if $\zeta(\delta_0, \delta_1) > \zeta(\delta_1, \delta_2) < \zeta(\delta_2, \delta_3)$, we get $\{x_0, x_1, x_2, x_3, \ldots\} \in [X]^r \cap P(1,0)$ or $\{x_1, x_2, x_3, x_4, \ldots\} \in [X]^r \cap P(0,1)$, giving a contradiction with (ii) or (i), respectively. ■

Proof (continued).

Now let such $s \leq 4$ be maximal (note $s \geq 3$ always holds) and define $x = x_{s-3}$, $y = x_{s-2}$ and $z = x_{s-1}$. Note that $\delta(x, y) \neq \delta(y, z)$, hence either (a) $\delta(x, y) > \delta(y, z)$ or (b) $\delta(x, y) < \delta(y, z)$. Then by maximality of s, for all $z \leq^* z_0 <^* z_1$, either

$$\text{(a) } \textbf{not } \delta(x, y) > \delta(y, z_0) < \delta(z_0, z_1), \text{ or}$$
$$\text{(b) } \textbf{not } \delta(x, y) < \delta(y, z_0) > \delta(z_0, z_1).$$

We show case (a) is impossible. For suppose otherwise, then for all $z_0 \in X$ with $z <^* z_0$ we have

$$\delta(y, z) > \delta(z, z_0) = \delta(y, z_0).$$

Picking an $<^*$-increasing sequence $\langle z_n \mid n < \omega \rangle$ gives us

$$\delta(y, z_0) > \delta(y, z_1) > \delta(y, z_2) > \dots,$$

which is a contradiction.

Proof (continued).

So, assume (b) holds. Let $z_0, z_1, z_2 \in X$ be arbitrary such that $z <^* z_0 <^* z_1 <^* z_2$. Then firstly, $\delta(x, y) < \delta(y, z) < \delta(z, z_0)$, hence $\delta(x, y) < \delta(y, z_0) = \delta(y, z)$. As $\delta(x, y) < \delta(y, z_0)$, it must be that $\delta(x, y) < \delta(y, z_0) < \delta(z_0, z_1)$.
Then $\delta(x, z_0) = \delta(x, y)$ and so $\delta(x, z_0) < \delta(z_0, z_1)$. Therefore, in view of the maximality of s,

$$\delta(z_0, z_1) < \delta(z_1, z_2).$$

Define $Y = \{z' \in X \mid z <^* z'\}$, we showed that $[Y]^r \subseteq P_0$ and clearly $|Y| = |X|$. $\square$

Theorem (Negative Stepping-Up Lemma, [2])

Suppose $r \geq 3$ and that κ and λ are infinite cardinals. Assume $\kappa \not\to (\lambda)^r_2$. Then $2^\kappa \not\to (\lambda)^{r+1}_2$.

Proof.

Let $[\kappa]^r = I_0 \cup\!\!\!\cdot\; I_1$ be the partition witnessing $\kappa \not\to (\lambda)^r_2$.
Define a partition $[2^\kappa]^{r+1} = J_0 \cup\!\!\!\cdot\; J_1$ by

$$J_1 = K(0,1) \cup P(0,1) \cup I_1^*,$$

and

$$J_0 = [2^\kappa]^{r+1} \setminus J_1.$$

Suppose there is $X \subseteq 2^\kappa$ such that $|X| = \lambda$ and $[X]^{r+1} \subseteq J_0$. Then $[X]^{r+1} \cap K(0,1) = \varnothing$, hence there is by the previous lemma some $Y \subseteq X$ with $|Y| = \lambda$ and $[Y]^{r+1} \subseteq K_0$ or $[Y]^{r+1} \subseteq K_1$. By the other lemma, there is $Z \subseteq Y$ with $|Z| = \lambda$ and $[Z]^{r+1} \subseteq P_0$. But this means $[Z]^{r+1} \subseteq I_0^*$. Using another lemma, we find a homogeneous set of size λ in I_0, a contradiction.

Proof (continued).

Similarly, suppose $X \subseteq 2^\kappa$ such that $|X| = \lambda$ and $[X]^{r+1} \subseteq J_1$. Then $[X]^{r+1} \subseteq K \cup K(0,1)$, and thus $[X]^{r+1} \cap K(1,0) = \varnothing$. This gives some $Y \subseteq X$ with $[Y]^{r+1} \subseteq K_0$ or $[Y]^{r+1} \subseteq K_1$ and $|Y| = \lambda$. Then $[Y]^{r+1} \subseteq P(0,1) \cup P_0$, hence $[Y]^{r+1} \cap P(1,0) = \varnothing$. Thus there is $Z \subseteq Y$ with $[Z]^{r+1} \subseteq P_0$. Therefore $[Z]^{r+1} \subseteq I_1^*$ and so we find a homogeneous set of size λ in I_1, a contradiction.
Therefore $2^\kappa \not\to (\lambda)_2^{r+1}$. □

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[3] Albin L. Jones. *A short proof of a partition relation for triples*. The Electronic Journal of Combinatorics, 7:R24, 2000.