Negative step-up results for partition relations
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Colourings

• \([S]^r = \{A \subseteq S : |A| = r\}\).

• A \textit{k-colouring} of \([S]^r\) is a function \(f : [S]^r \rightarrow k\).

• Given \(f : [S]^r \rightarrow k\), a set \(H \subseteq S\) is \(i\)-\textit{homogeneous for} \(f\) if \(f \upharpoonright [H]^r\) is constant with colour \(i \in k\).

Definition (Arrow notation)

Let \(\alpha\) and \(\beta_i\) be order-types for all \(i < m\), where \(m\) is a cardinal and let \(r \in \mathbb{N}\). We write

\[ \alpha \rightarrow (\beta_i)_{i < m}^r, \]

if for all sets \(S\) with \(\text{otp} S = \alpha\) and every \(m\)-colouring \(f : [S]^r \rightarrow m\) there exists a \(i\)-homogeneous set \(H \subseteq S\) with \(\text{otp} H = \beta_i\).

• \(\alpha\) is the \textit{resource},

• \(\beta_i\) are the \textit{goals},

• \(r\) is the \textit{exponent}, and

• \(m\) is the \textit{colour set} or \textit{colour cardinal}.
Positive Step-Up Lemma

Theorem (Theorem 39, [1])

For all infinite cardinals $\kappa$, finite $r$, any cardinal $m$ and any ordinal $\lambda$, if $\kappa \to (\lambda)_m^r$, then $(2^{<\kappa})^+ \to (\lambda + 1)_m^{r+1}$.

Theorem

For any $r, k \in \mathbb{N}$, $\omega_1 \to (\omega + 1)_k^r$.

Theorem

Let $r, k \in \mathbb{N}$. Then $(2^{\aleph_0})^+ \to (\omega + 2)_k^r$.

Theorem (Erdős-Rado Theorem)

For any infinite cardinal $\kappa$, $\beth_n(\kappa)^+ \to (\kappa^+)^{n+1}_{\kappa}$.
Pattern of partition relations

**Theorem**

For any $r, k \in \mathbb{N}$, $\omega_1 \rightarrow (\omega + 1)_k^r$.

**Theorem**

Let $r, k \in \mathbb{N}$. Then $(2^{\aleph_0})^+ \rightarrow (\omega + 2)_k^r$.

- $\square^+_0 \rightarrow (\omega + 1)_k^r$.
- $\square^+_1 \rightarrow (\omega + 2)_k^r$.
- $\square^+_2 \rightarrow (\omega + 3)_k^r$.
- etc...
Pattern of partition relations

- $\mathcal{P}_0^+ \rightarrow (\omega + 1)^r_k$.
- $\mathcal{P}_0^+ \not\rightarrow (\omega + 2)^3_2$.
- $\mathcal{P}_1^+ \rightarrow (\omega + 2)^r_k$.
- $\mathcal{P}_1^+ \not\rightarrow (\omega + 3)^4_2$.
- $\mathcal{P}_2^+ \rightarrow (\omega + 3)^r_k$.
- $\mathcal{P}_2^+ \not\rightarrow (\omega + 4)^5_2$?
Theorem (Lemma 4, [1])

Let $\alpha_0, \alpha_1, \beta_0, \beta_1$ be linear order-types and $2 \leq r < \omega$. Assume that $|\alpha_0| = |\alpha_1|$ and $\beta_0, \beta_1^* \not\leq \alpha_0$. Then

$$\alpha_1 \not\rightarrow (\beta_0, \beta_1, (r + 1)r! - 2)_r.$$

Proof.

Throughout we may assume $|\beta_0|, |\beta_1| \geq \aleph_0$. Let $S$ be a set such that $\text{otp}(S, <) = \alpha_1$. As $|S| = |\alpha_1| = |\alpha_0|$, there is an ordering $\ll$ on $S$ such that $\text{otp}(S, \ll) = \alpha_0$. Given any $X \in [S]^r$, we can index the elements in $X$ such that $X = \{x_0 < x_1 < \ldots < x_{r-1}\}$. There is a unique permutation $\pi : r \rightarrow r$ such that $x_{\pi(0)} \ll x_{\pi(1)} \ll \ldots \ll x_{\pi(r-1)}$. Note that there are precisely $r!$ permutations of $r$. Fix an enumeration $\langle \pi_n \mid n < r! \rangle$ of permutations of $r$, where $\pi_0$ is the identity, and $\pi_1 = \pi_0^*$. 
Proof (continued).

Define the $r!$-colouring $f : [S]^r \to r! : \{ x_0 < x_1 < \ldots < x_{r-1} \} \mapsto n$ where $\pi_n$ is such that $x_{\pi_n(0)} \ll x_{\pi_n(1)} \ll \ldots \ll x_{\pi_n(r-1)}$. Suppose there is a $n$-homogeneous set $H$ for $f$ with $\text{otp}(H, <) = \beta_n$, there are three cases that we consider.

Case $n = 0$. Then $\text{otp}(H, <) = \beta_0$ and $f \upharpoonright [H]^r \equiv 0$. As $\pi_0$ is the identity and $r \geq 2$, we have in particular for any $x, y \in H$ that $x < y \iff x \ll y$. This means $\beta_0 = \text{otp}(H, <) = \text{otp}(H, \ll) \leq \text{otp}(S, \ll) = \alpha_0$, which is a contradiction.

Case $n = 1$. Then $\text{otp}(H, <) = \beta_1$ and $f \upharpoonright [H]^r \equiv 1$. In this case $\pi_1 = \pi_0^*$, which means for any $x, y \in H$ we have $x < y \iff y \ll x$. Therefore $\beta_1^* = \text{otp}(H, <)^* = \text{otp}(H, \ll) \leq \text{otp}(S, \ll) = \alpha_0$, again a contradiction.
Case $n \geq 2$. Then $\text{otp}(H, <) = r + 1$ and $f \upharpoonright [H]^r \equiv n$. In particular $\pi_n \neq \pi_0$ and $\pi_n \neq \pi_1$. Write $H = \{x_0 < x_1 < \ldots < x_{r-1} < x_r\}$ and define $y_k = x_{k+1}$. Then

$$x_{\pi_n(0)} \ll x_{\pi_n(1)} \ll \ldots \ll x_{\pi_n(r-1)},$$

$$y_{\pi_n(0)} \ll y_{\pi_n(1)} \ll \ldots \ll y_{\pi_n(r-1)}.$$

Suppose $x_0 \ll x_1$, then $x_{\pi_n(0)-1} < x_{\pi_n(1)-1}$ and so $y_0 \ll y_1$. This gives $x_1 \ll x_2$. Repeating this argument gives that $x_0 \ll x_1 \ll \ldots \ll x_{r-1}$, and hence $\pi_n = \pi_0$, which is a contradiction. If, on the other hand, we assume $x_1 \ll x_0$, then using an analogous argument, we get $x_{r-1} \ll \ldots \ll x_1 \ll x_0$, i.e. $\pi_n = \pi_1$, which is also a contradiction. We conclude that such a homogeneous set $H$ cannot exist, and this concludes the proof.
Theorem (Lemma 5, [1])

Let $\alpha$ be an ordinal and $r < \omega$ and $m$ any cardinal. Let $\gamma_n$ be ordinals for all $n < m$ such that $\beta \not\rightarrow (\gamma_n)^r_{n<m}$ for all $\beta < \alpha$. Then $\alpha \not\rightarrow (\gamma_n + 1)^{r+1}_{n<m}$.

Proof.

Let $S$ be a set such that $\text{otp}(S, <) = \alpha$. For every $x \in S$, define $l_x = \{y \in S \mid y < x\}$, then $\text{otp}(l_x, <) = \beta < \alpha$, for some $\beta$. By assumption there is some colouring $f_x : [l_x]^r \rightarrow k$ such that there is no $n$-homogeneous set $H$ of order-type $\gamma_n$.

Define $f : [S]^{r+1} \rightarrow k$ by

$$\{x_0 < x_1 \ldots < x_{r-1} < x_r\} \mapsto f_{x_r}(\{x_0 < x_1 \ldots < x_{r-1}\}).$$

If there is an $n$-homogeneous set $H = \{h_i \mid i < \gamma_n + 1\} \subseteq S$ for $f$ with $\text{otp}(H, <) = \gamma_n + 1$, then the set $\{h_i \mid i < \gamma_n\}$ is $n$-homogeneous for $f_{h_{\gamma_n}}$ with order-type $\gamma_n$. This is a contradiction and hence the proof is concluded.
The first negative relation

**Theorem (Theorem 41, [1])**
\[ \omega_1 \not
\rightarrow (\omega + 2)_2^3. \]

**Proof.**
Clearly, \( \omega + 1, \omega^* \not
\leq \omega \). By Lemma 4, we have \( \beta \not
\rightarrow (\omega + 1, \omega)_2^2 \) for all countable ordinals \( \beta \). This relation holds for all \( \beta \prec \omega_1 \) and thus by Lemma 5, \( \omega_1 \not
\rightarrow (\omega + 2, \omega + 1)_2^3. \)
Recapitulate

- $\mathcal{B}_0^+ \rightarrow (\omega + 1)_k^r$.
- $\mathcal{B}_0^+ \not\rightarrow (\omega + 2)_2^3$.
- $\mathcal{B}_1^+ \rightarrow (\omega + 2)_k^r$.
- $\mathcal{B}_2^+ \rightarrow (\omega + 3)_k^r$.
- $\mathcal{B}_1^+ \not\rightarrow (\omega + 3)_2^4$. 
Definition

Given distinct $f, g \in 2^\kappa$, we define the discrepancy $\delta$ as

$$\delta(f, g) = \min\{\xi < \kappa \mid f(\xi) \neq g(\xi)\}.$$ 

If $f = g$, we simply let $\delta(f, g) = \kappa$.

Observation

If $\delta(f, g) < \delta(g, h)$, then $\delta(f, g) = \delta(f, h)$.

Remark

Let $\prec$ denote the lexicographic ordering on $2^\kappa$. If $f, g, h \in 2^\kappa$ are such that $f \prec g \prec h$, then $\delta(f, g) \neq \delta(g, h)$. Else, $f(\xi) < g(\xi) < h(\xi)$ for some $\xi < \kappa$. 
Theorem (Albin L. Jones (2000), [3])

Let $\alpha$ be a linear order-type and let $\kappa$ be an infinite cardinal. If $\alpha \not\to (\omega)_{2\kappa},$ then $\alpha \not\to (\kappa + 2, \omega)^3_2.$

Proof.

Let $e: \alpha \to 2^{\kappa}$ be a witness of $\alpha \not\to (\omega)_{2\kappa}.$ As $\omega$ is regular, it follows for every $B \in [\alpha]^\omega$ there is $C \in [B]^\omega$ such that $e \upharpoonright C$ is injective. Define the partition $f: [\alpha]^2 \to \kappa + 1$ by $\{x, y\} \mapsto \delta(e(x), e(y)).$

Define the partition of triples $g: [\alpha]^3 \to 2$ s.t. for $x < y < z,$

$$g\{x, y, z\} = \begin{cases} 0 & \text{if } e \text{ is injective on } \{x, y, z\} \text{ and } f\{x, y\} < f\{y, z\}, \text{ and} \\ 1 & \text{if } e \text{ is not injective on } \{x, y, z\} \text{ or } f\{x, y\} \geq f\{y, z\}. \end{cases}$$

We show that $g$ is the partition which proves $\alpha \not\to (\kappa + 2, \omega)^3_2.$
Claim

*There is no 0-homogeneous $H \subseteq \alpha$ for $g$ with $otp\ H = \kappa + 2$.***

Proof of claim.

Suppose such $H = \{h_\gamma \mid \gamma < \kappa + 2\}$ exists. We observe immediately that $e \upharpoonright H$ is injective. In particular, $e(h_\kappa) \neq e(h_{\kappa+1})$ and hence $f\{h_\kappa, h_{\kappa+1}\} = \delta(e(h_\kappa), e(h_{\kappa+1})) = \xi < \kappa$. For any $\mu < \nu < \kappa$ we have $f\{h_\mu, h_\nu\} < f\{h_\nu, h_\kappa\}$, and by the observation:

$$f\{h_\mu, h_\kappa\} = f\{h_\mu, h_\nu\} < f\{h_\nu, h_\kappa\}.$$  

Note that $f\{h_\mu, h_\kappa\} < f\{h_\kappa, h_{\kappa+1}\} = \xi < \kappa$. Hence, the sequence $\langle f\{h_\mu, h_\kappa\} \mid \mu < \kappa \rangle$ is a strictly increasing sequence of length $\kappa$ of ordinals below $\xi$, which gives a contradiction.  

$\blacksquare$
Claim

There is no 1-homogeneous $H \subseteq \alpha$ for $g$ with $\text{otp } H = \omega$.

Proof of claim.

Again, for sake of contradiction assume such $H \in [\alpha]^{\omega}$ exists. By the remark above there is $B \in [H]^{\omega}$ such that $e \upharpoonright B$ is injective. Consider the colouring $h: [B]^3 \to 2$ by

$$h\{x < y < z\} = \begin{cases} 0 & \text{ if } f\{x, y\} > f\{y, z\}, \text{ and} \\ 1 & \text{ if } f\{x, y\} = f\{y, z\}. \end{cases}$$

By definition of $g$ and since $B$ is 1-homogeneous for $g$, the colouring $h$ is well-defined. Now, by a weak version of Ramsey's Theorem, $\omega \to (\omega, 4)^3_2$. Hence, either

(a) there is $C \in [B]^{\omega}$ such that $h \upharpoonright [C]^3 \equiv 0$, or

(b) there is $D \in [B]^{4}$ such that $h \upharpoonright [D]^3 \equiv 1$.

If (a) holds, then $\langle f\{c_n, c_{n+1}\} \mid n \in \omega \rangle$, where $C = \{c_0 < c_1 < \ldots\}$, is a strictly decreasing sequence of ordinals of length $\omega$, which is a contradiction.
Proof of claim (continued).

Alternatively, if (b) holds and such $D = \{x < y < z < w\}$ exists, then $f\{x, y\} = f\{y, z\} = f\{x, z\}$. This gives us three pairwise distinct functions $e(x), e(y), e(z) \in 2^{\kappa}$ such that they are pairwise different at some point $\xi < \kappa$, which is not possible since these functions map to 2. Hence we reach a contradiction.
The second negative relation

Proof (continued).

We conclude that

\[ \alpha \not\to (\omega)_{2^\kappa}^1 \implies \alpha \not\to (\kappa + 2, \omega)^3. \]

Theorem

\[ \bigboxplus_1^+ \not\to (\omega + 3, \omega + 1)^4. \]

Proof.

Let \( \beta < \bigboxplus_1^+ \), then \( |\beta| \leq \bigboxplus_1 = 2^{<\omega_0} \). Define the partition

\[ f: \beta \to \beta: \gamma \mapsto \gamma. \]

Then \( f \) witnesses \( \beta \not\to (\omega)^1_\beta \). Therefore \( \beta \not\to (\omega)^1_{2^{<\omega_0}}. \)

By Jones's lemma, we have \( \beta \not\to (\omega + 2, \omega)^3. \)

Then by Lemma 4 in [1], we obtain the desired result.

David de Graaf (UvA)
Does the pattern continue?

- $\varpi^+_0 \rightarrow (\omega + 1)_k^r$.
- $\varpi^+_0 \not\rightarrow (\omega + 2)_2^3$.
- $\varpi^+_1 \rightarrow (\omega + 2)_k^r$.
- $\varpi^+_1 \not\rightarrow (\omega + 3)_2^4$.
- $\varpi^+_2 \rightarrow (\omega + 3)_k^r$.
- $\varpi^+_2 \not\rightarrow (\omega + 4)_2^5$?
The converse: $\kappa \not\rightarrow (\lambda)_k^\kappa \implies (2^{<\kappa})^+ \not\rightarrow (\lambda + 1)_k^\kappa$.

Assume $2^{\aleph_0} = \aleph_2$ and $2^{\aleph_1} = 2^{\aleph_2} = \aleph_3$. Then $\beth_2^+ = (2^{2^{\aleph_0}})^+ = \aleph_4$ and $(2^{<\aleph_2})^+ = \aleph_4$.

- Sierpiński: $\aleph_2 = 2^{\aleph_0} \not\rightarrow (\aleph_1)^2$
- Erdős-Rado: $\aleph_4 = \beth_2^+ \rightarrow (\aleph_1 + 1)^3$. 
The idea

- We want to show $\kappa \not

\to (\alpha)_m \implies 2^\kappa \not

\to (\alpha + 1)_{m+1}$.
- We will show $\kappa \not

\to (\lambda)_m \implies 2^\kappa \not

\to (\lambda)_{m+1}$.
- Given a partition $\langle I_\xi \mid \xi < m \rangle$ witnessing $\kappa \not

\to (\lambda)_m$.
- Want to create partition $\langle I^*_\xi \mid \xi < m \rangle$ witnessing $2^\kappa \not

\to (\lambda)_{m+1}$.
- The partition will (unfortunately) only work if $\lambda$ is a cardinal.
Let $r \geq 3$ and $u \in [2^\kappa]^r$. Write $u = \{x_0 <^* x_1 <^* \ldots <^* x_{r-1}\}$ and define

- $\eta(u) = (\eta(x_0, x_1), \eta(x_1, x_2), \ldots, \eta(x_{r-2}, x_{r-1}))$,

where $\eta(x, y) = 0$ if $x <^* y \iff x \prec y$, and $\eta(x, y) = 1$ otherwise.

Also given $s \leq r - 1$ and $k_0, k_1, \ldots, k_{s-1} \in 2$, define

- $K(k_0, k_1, \ldots, k_{s-1}) = \{u \in [2^\kappa]^r \mid \eta(u) \upharpoonright s = (k_0, k_1, \ldots, k_{s-1})\}$.

- $K_0 = K(0, 0, \ldots, 0)$ and $K_1 = K(1, 1, \ldots, 1)$.

- $K = K_0 \cup K_1$. 
Examples

Given $u = \{x_0 <^* x_1 <^* \ldots <^* x_{r-1}\}$.

$$u \in K(0, 1) \iff \eta(x_0, x_1) = 0 \text{ and } \eta(x_1, x_2) = 1$$

$$\iff x_0 \prec x_1 \succ x_2.$$

- $u \in K_0 \iff x_0 \prec x_1 \prec \ldots \prec x_{r-1}$
- $u \in K_1 \iff x_0 \succ x_1 \succ \ldots \succ x_{r-1}$
More definitions

Let \( r \geq 4 \) and \( u \in K \). Write \( u = \{ x_0 <^* x_1 <^* \ldots <^* x_{r-1} \} \) and define \( \delta_s = \delta(x_s, x_{s+1}) \) for \( s \leq r - 2 \). Define

- \( \zeta(\delta(u)) = (\zeta(\delta_0, \delta_1), \zeta(\delta_1, \delta_2), \ldots, \zeta(\delta_{r-3}, \delta_{r-2})) \),

where \( \zeta(\delta_s, \delta_{s+1}) = 0 \) if \( \delta_s < \delta_{s+1} \), and \( \zeta(\delta_s, \delta_{s+1}) = 1 \) if \( \delta_s > \delta_{s+1} \).

Also given \( s \leq r - 2 \) and \( k_0, k_1, \ldots, k_{s-1} \in 2 \), define

- \( P(k_0, k_1, \ldots, k_{s-1}) = \{ u \in K \mid \zeta(\delta(u)) \upharpoonright s = (k_0, k_1, \ldots, k_{s-1}) \} \).
- \( P_0 = P(0, 0, \ldots, 0) \) and \( P_1 = P(1, 1, \ldots, 1) \).
- \( P = P_0 \cup P_1 \).
Given \( u = \{x_0 <^* x_1 <^* \ldots <^* x_{r-1}\} \).

\[
u \in P(0, 1) \iff \zeta(\delta_0, \delta_1) = 0 \text{ and } \zeta(\delta_1, \delta_2) = 1
\iff \delta_0 < \delta_1 > \delta_2
\iff \delta(x_0, x_1) < \delta(x_1, x_2) > \delta(x_2, x_3).
\]

- \( u \in P_0 \iff \delta_0 < \delta_1 < \ldots < \delta_{r-2} \)
- \( u \in P_1 \iff \delta_0 > \delta_1 > \ldots > \delta_{r-2} \)
Lemma (Lemma 23.12, [2])

Let $r \geq 3$, let $\kappa$ be a cardinal. Let $I \subseteq [\kappa]^{r-1}$ and put

$$I^* = \{ u \in P_0 \mid \delta(u) \in I \}. \tag{1}$$

Assume that $[H]^r \subseteq I^*$ for some $H \neq \emptyset$ where by assumption $\text{otp}(H, <^*) = \alpha$. Then there is $X \subseteq \kappa$ with $\text{otp}(X, <) = \alpha^-$ such that $[X]^{r-1} \subseteq I$.

Proof.

We may assume that $|H| \geq r$ and write $H = \{ h_\gamma \mid \gamma < \alpha \}$ where $\alpha = \text{otp}(H, <^*)$. (Recall that $<^*$ is a fixed well-order on $2^\kappa$). For ordinals $\gamma$ such that $\gamma + 1 < \alpha$ we let

$$\delta_\gamma = \delta(h_\gamma, h_{\gamma+1}).$$

Define

$$X = \{ \delta_\gamma \mid \gamma + 1 < \alpha \}.$$
Proof (continued).

First we show that \( \text{otp}(X, <) = \alpha^- \). It obviously suffices to show for all \( \gamma < \gamma' < \alpha^- \) that \( \delta_\gamma < \delta_{\gamma'} \). By the assumption \([H]' \subseteq I^* \subseteq P_0\), it follows that

\[
\zeta(\delta(\{h_\gamma, h_{\gamma+1}, h_{\gamma'}\})) = \zeta(\delta(h_\gamma, h_{\gamma+1}), \delta(h_{\gamma+1}, h_{\gamma'})) = 0.
\]

Also

\[
\zeta(\delta(\{h_{\gamma+1}, h_{\gamma'}, h_{\gamma'+1}\})) = \zeta(\delta(h_{\gamma+1}, h_{\gamma'}), \delta(h_{\gamma'}, h_{\gamma'+1})) = 0.
\]

In other words, \( \delta_\gamma < \delta(h_{\gamma+1}, h_{\gamma'}) < \delta_{\gamma'} \). Note that we assumed \( \gamma + 1 < \gamma' \), because if \( \gamma + 1 = \gamma' \), we could just leave out the term \( \delta(h_{\gamma+1}, h_{\gamma'}) \). In particular, we obtain \( \delta_\gamma < \delta_{\gamma'} \), showing that \( \text{otp}(X, <) = \alpha^- \).
Proof (continued).

It rests to show that $[X]^{r-1} \subseteq I$. Given $\xi_0 < \ldots < \xi_{r-2} < \alpha^-$, we want to show $\{\delta_{\xi_0} < \ldots < \delta_{\xi_{r-2}}\} \in I$. Suppose that $\xi_i + 1 < \xi_{i+1}$. As $[H]^r \subseteq P_0$, we have $\delta(h_{\xi_i}, h_{\xi_{i+1}}) < \delta(h_{\xi_{i+1}}, h_{\xi_{i+1}})$ and hence, $\delta(h_{\xi_i}, h_{\xi_{i+1}}) = \delta(h_{\xi_i}, h_{\xi_{i+1}})$. If $\xi_i + 1 = \xi_{i+1}$, then $\delta(h_{\xi_i}, h_{\xi_{i+1}}) = \delta(h_{\xi_i}, h_{\xi_{i+1}})$ obviously holds as well. Now, writing $\xi_{r-1} = \xi_{r-2} + 1$, we obtain

$$\{\delta_{\xi_i} \mid i < r - 1\} = \{\delta(h_{\xi_i}, h_{\xi_{i+1}}) \mid i < r - 1\}$$
$$= \{\delta(h_{\xi_i}, h_{\xi_{i+1}}) \mid i < r - 1\}$$
$$= \delta(\{h_{\xi_i} \mid i < r\}).$$

As $\{h_{\xi_i} \mid i < r\} \in [H]^r \subseteq I^*$, we have by definition of $I^*$ that $\{\delta_{\xi_i} \mid i < r - 1\} \in I$. This gives us $[X]^{r-1} \subseteq I$, which is what we wanted to show. \qed

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Lemma (Lemma 23.5, [2])

Let $X \subseteq 2^\kappa$ and assume $|X| \geq \aleph_0$. Assume that (i) $[X]^r \cap K(0, 1) = \emptyset$ or (ii) $[X]^r \cap K(1, 0) = \emptyset$. Then there is a set $Y \subseteq X$ with $|Y| = |X|$ such that $[Y]^r \subseteq K_0$ or $[Y]^r \subseteq K_1$.

Proof.

Write $\lambda = |X|$ and we may assume $\otp(X, <^*) = \lambda$. Assume that no such $Y$ exists.

Claim

There are elements $x_0 <^* x_1 <^* x_2 <^* x_3$ such that $x_0 \prec x_1 \succ x_2 \prec x_3$.

If the claim is proven, then there is $\{x_0, x_1, x_2, \ldots\} \in [X]^r \cap K(0, 1)$ and $\{x_1, x_2, x_3, \ldots\} \in [X]^r \cap K(1, 0)$, contradicting (i) or (ii), respectively, which gives the contradiction. Hence such $Y$ exists.
Claim

There are elements \( x_0 <^* x_1 <^* x_2 <^* x_3 \) such that \( x_0 \prec x_1 \succ x_2 \prec x_3 \).

Proof of claim.

For every \( x \in X \) there are \( y, z \in X \) and \( y', z' \in X \) such that

\[
\begin{align*}
  x &\leq^* y <^* z \text{ and } y \prec z, \\
  x &\leq^* y' <^* z' \text{ and } y' \succ z'.
\end{align*}
\]

(2) (3)

Suppose not and let \( x \in X \) be a counterexample, the set \( Y = \{ x' \in X \mid x \leq^* x' \} \) has cardinality \( \lambda \) and is contained in either \( K_0 \) or \( K_1 \), which is a contradiction.

Now let \( x_0, z_1 \in X \) with \( x_0 <^* z_1 \) and \( x_0 \prec z_1 \). Then let \( y_1, z_2 \in X \) with \( z_1 \leq^* y_1 <^* z_2 \) with \( y_1 \succ z_2 \). Define \( x_1 = \max_\prec \{ y_1, z_1 \} \), then \( x_0 <^* x_1 \) and \( x_0 \prec x_1 \). Also, \( x_1 \succ z_2 \).

Pick \( y_2, z_3 \in X \) with \( z_2 \leq^* y_2 <^* x_3 \) with \( y_2 \prec x_3 \). Let \( x_2 = \min_\prec \{ y_2, z_2 \} \).

Then \( x_1 <^* x_2 \) and \( x_1 \succ x_2 \). Also, \( x_2 \prec x_3 \). This proves the claim.
Lemma (Lemma 23.9, [2])

Let \( r \geq 4 \), let \( X \subseteq 2^\kappa \) such that \( |X| \geq \aleph_0 \). Suppose \([X]^r \subseteq K_0\) or \([X]^r \subseteq K_1\). Assume (i) \([X]^r \cap P(0, 1) = \emptyset\) or (ii) \([X]^r \cap P(1, 0) = \emptyset\). Then there exists \( Y \subseteq X \) with \( |Y| = |X| \) such that \([Y]^r \subseteq P_0\).

Claim

Suppose \( x_0 <^* x_1 <^* \ldots <^* x_{s-1} \) are such that

\[
\zeta(\delta_i, \delta_{i+1}) \neq \zeta(\delta_{i+1}, \delta_{i+2}),
\]

(4)

for all \( i \leq s - 4 \). Then \( s \leq 4 \).

Proof of claim.

Suppose \( s \geq 5 \) and \( x_0 <^* x_1 <^* x_2 <^* x_3 <^* x_4 \) constitutes a counterexample. If \( \zeta(\delta_0, \delta_1) < \zeta(\delta_1, \delta_2) > \zeta(\delta_2, \delta_3) \), then \( \{x_0, x_1, x_2, x_3, \ldots\} \in [X]^r \cap P(0, 1) \) or \( \{x_1, x_2, x_3, x_4, \ldots\} \in [X]^r \cap P(1, 0) \), giving a contradiction with (i) or (ii), respectively.

Similarly, if \( \zeta(\delta_0, \delta_1) > \zeta(\delta_1, \delta_2) < \zeta(\delta_2, \delta_3) \), we get \( \{x_0, x_1, x_2, x_3, \ldots\} \in [X]^r \cap P(1, 0) \) or \( \{x_1, x_2, x_3, x_4, \ldots\} \in [X]^r \cap P(0, 1) \), giving a contradiction with (ii) or (i), respectively. ■
Proof (continued).

Now let such $s \leq 4$ be maximal (note $s \geq 3$ always holds) and define $x = x_{s-3}$, $y = x_{s-2}$ and $z = x_{s-1}$. Note that $\delta(x, y) \neq \delta(y, z)$, hence either (a) $\delta(x, y) > \delta(y, z)$ or (b) $\delta(x, y) < \delta(y, z)$. Then by maximality of $s$, for all $z \leq^* z_0 <^* z_1$, either

(a) **not** $\delta(x, y) > \delta(y, z_0) < \delta(z_0, z_1)$, or
(b) **not** $\delta(x, y) < \delta(y, z_0) > \delta(z_0, z_1)$.

We show case (a) is impossible. For suppose otherwise, then for all $z_0 \in X$ with $z <^* z_0$ we have

$$\delta(y, z) > \delta(z, z_0) = \delta(y, z_0).$$

Picking an $<^*$-increasing sequence $\langle z_n \mid n < \omega \rangle$ gives us

$$\delta(y, z_0) > \delta(y, z_1) > \delta(y, z_2) > \ldots,$$

which is a contradiction.
Proof (continued).

So, assume (b) holds. Let $z_0, z_1, z_2 \in X$ be arbitrary such that $z <^* z_0 <^* z_1 <^* z_2$. Then firstly, $\delta(x, y) < \delta(y, z) < \delta(z, z_0)$, hence $\delta(x, y) < \delta(y, z_0) = \delta(y, z)$. As $\delta(x, y) < \delta(y, z_0)$, it must be that $\delta(x, y) < \delta(y, z_0) < \delta(z_0, z_1)$.

Then $\delta(x, z_0) = \delta(x, y)$ and so $\delta(x, z_0) < \delta(z_0, z_1)$. Therefore, in view of the maximality of $s$, $\delta(z_0, z_1) < \delta(z_1, z_2)$.

Define $Y = \{z' \in X \mid z <^* z'\}$, we showed that $[Y]^r \subseteq P_0$ and clearly $|Y| = |X|$.
Theorem (Negative Stepping-Up Lemma, [2])

Suppose $r \geq 3$ and that $\kappa$ and $\lambda$ are infinite cardinals. Assume $\kappa \nrightarrow (\lambda)^r_2$. Then $2^\kappa \nrightarrow (\lambda)^{r+1}_2$.

Proof.

Let $[\kappa]^r = I_0 \cup I_1$ be the partition witnessing $\kappa \nrightarrow (\lambda)^r_2$. Define a partition $[2^\kappa]^{r+1} = J_0 \cup J_1$ by

$$J_1 = K(0, 1) \cup P(0, 1) \cup I_1^*,$$

and

$$J_0 = [2^\kappa]^{r+1} \setminus J_1.$$

Suppose there is $X \subseteq 2^\kappa$ such that $|X| = \lambda$ and $[X]^{r+1} \subseteq J_0$. Then $[X]^{r+1} \cap K(0, 1) = \emptyset$, hence there is by the previous lemma some $Y \subseteq X$ with $|Y| = \lambda$ and $[Y]^{r+1} \subseteq K_0$ or $[Y]^{r+1} \subseteq K_1$. By the other lemma, there is $Z \subseteq Y$ with $|Z| = \lambda$ and $[Z]^{r+1} \subseteq P_0$. But this means $[Z]^{r+1} \subseteq I_0^*$. Using another lemma, we find a homogeneous set of size $\lambda$ in $I_0$, a contradiction.
Proof (continued).

Similarly, suppose $X \subseteq 2^\kappa$ such that $|X| = \lambda$ and $[X]^{r+1} \subseteq J_1$. Then $[X]^{r+1} \subseteq K \cup K(0,1)$, and thus $[X]^{r+1} \cap K(1,0) = \emptyset$. This gives some $Y \subseteq X$ with $[Y]^{r+1} \subseteq K_0$ or $[Y]^{r+1} \subseteq K_1$ and $|Y| = \lambda$. Then $[Y]^{r+1} \subseteq P(0,1) \cup P_0$, hence $[Y]^{r+1} \cap P(1,0) = \emptyset$. Thus there is $Z \subseteq Y$ with $[Z]^{r+1} \subseteq P_0$. Therefore $[Z]^{r+1} \subseteq I_1^*$ and so we find a homogeneous set of size $\lambda$ in $I_1$, a contradiction. Therefore $2^\kappa \not\rightarrow (\lambda)^2_{r+1}$. 

\hfill \Box
