Separating Many Slalom Cardinals

Tristan van der Vlugt
Universität Hamburg

STiHAC Forschungsseminar Mathematische Logik

December 3, 2021

Main goal: generalise results known for the **classical** reals ${}^{\omega}2$ to the **generalised** reals ${}^{\kappa}2$.

Classically, for every cofinal $h \in {}^{\omega}\omega$ we can define what an h-slalom is. We can define cardinal characteristics $\mathfrak{b}^h_\omega(\in^*)$ and $\mathfrak{d}^h_\omega(\in^*)$. It can be proved that the choice of h does not matter: $\mathfrak{b}^h_\omega(\in^*) = \mathfrak{b}^g_\omega(\in^*)$ and $\mathfrak{d}^h_\omega(\in^*) = \mathfrak{d}^g_\omega(\in^*)$ for all h, g.

Generally, for inaccessible κ, it was found that if $\mathrm{id} : \alpha \mapsto \alpha$ and $\mathrm{pow} : \alpha \mapsto 2^{|\alpha|}$, then $\mathfrak{d}^{\mathrm{pow}}_\kappa(\in^*) < \mathfrak{d}^{\mathrm{id}}_\kappa(\in^*)$ is consistent.

In this talk we show that there is a sequence $\langle h_\alpha \in {}^{\kappa}\kappa \mid \alpha < \kappa \rangle$ such that $\mathfrak{d}^{h_{\alpha'}}_\kappa(\in^*) < \mathfrak{d}^{h_\alpha}_\kappa(\in^*)$ is consistent for all $\alpha < \alpha'$.

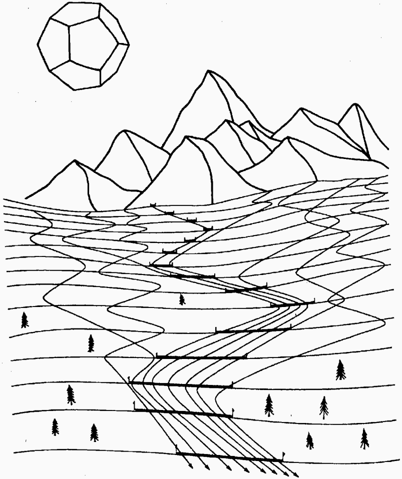
Contents

Let κ be regular strong limit and $h \in {}^{\kappa}\kappa$ be an increasing cofinal cardinal function.

An h-**slalom** is any function $\varphi : \kappa \to [\kappa]^{<\kappa}$ such that $|\varphi(\alpha)| = h(\alpha)$ for all $\alpha \in \kappa$.

For $f \in {}^{\kappa}\kappa$, we say $f \in^* \varphi$, or f is **localised** by φ, if there exists some $\xi < \kappa$ such that $f(\alpha) \in \varphi(\alpha)$ for all $\alpha \in [\xi, \kappa)$.

We will let Loc_h be the set of h-slaloms.



[Bartoszyński, 1987]

We define the following cardinal characteristics:

$$\mathfrak{b}^h_\kappa(\in^*) = \min\{|B| \mid B \subseteq {}^\kappa\kappa \text{ and } \forall\varphi \in \mathrm{Loc}_h \exists f \in B(f \notin^* \varphi)\},$$
$$\mathfrak{d}^h_\kappa(\in^*) = \min\{|D| \mid D \subseteq \mathrm{Loc}_h \text{ and } \forall f \in {}^\kappa\kappa \exists\varphi \in D(f \in^* \varphi)\}.$$

Proposition *[Brendle et al., 2018] sections 4.3 & 4.4*

$\kappa^+ \leq \mathfrak{b}^h_\kappa(\in^*) \leq \mathfrak{d}^h_\kappa(\in^*) \leq 2^\kappa$, and all relations can consistently be strict inequalities. □

Let $\mathcal{N}$ be the ideal of sets of reals with Lebesgue measure 0.

$$\mathrm{add}(\mathcal{N}) = \min\left\{|\mathcal{A}| \;\middle|\; \mathcal{A} \subseteq \mathcal{N} \text{ and } \bigcup\mathcal{A} \notin \mathcal{N}\right\},$$
$$\mathrm{cof}(\mathcal{N}) = \min\left\{|\mathcal{C}| \;\middle|\; \mathcal{C} \subseteq \mathcal{N} \text{ and } \forall N \in \mathcal{N} \exists C \in \mathcal{C}(N \subseteq C)\right\}.$$

Proposition *[Bartoszyński, 1987] or [Bartoszyński and Judah, 1995]*

$\mathfrak{b}_\omega(\in^*) = \mathrm{add}(\mathcal{N})$ and $\mathfrak{d}_\omega(\in^*) = \mathrm{cof}(\mathcal{N})$ □

Theorem *[Bartoszyński, 1987] or [Blass, 2010] remark 5.15 (for $\kappa = \omega$)*
If $h, g \in {}^\kappa\kappa$ are continuous (i.e. $h(\gamma) = \bigcup_{\alpha<\gamma} h(\alpha)$ for limit γ) and unbounded, then $\mathfrak{d}^h_\kappa(\in^*) = \mathfrak{d}^g_\kappa(\in^*)$ and $\mathfrak{b}^h_\kappa(\in^*) = \mathfrak{b}^g_\kappa(\in^*)$.

Proof. Let $\langle \xi_\alpha \mid \xi \in \kappa \rangle$ enumerate a club s.t. $h(\alpha) \leq g(\xi_\alpha)$, and let $I_\alpha = [\xi_\alpha, \xi_{\alpha+1})$. Fix some bijections $\pi_\alpha : \kappa \rightarrowtail\!\!\!\!\!\twoheadrightarrow {}^{I_\alpha}\kappa$.

For any $f \in {}^\kappa\kappa$ let $f' : \alpha \mapsto \pi^{-1}_\alpha(f \restriction I_\alpha)$. For any $\varphi \in \mathrm{Loc}_h$ and $\xi \in I_\alpha$ let $\varphi'(\xi) \supseteq \{\pi_\alpha(i)(\xi) \mid i \in \varphi(\alpha)\}$ s.t. $|\varphi'(\xi)| = |g(\xi)|$.

If $f' \in^* \varphi$, let α be s.t. $f'(\alpha) \in \varphi(\alpha)$. Then $\pi_\alpha(f'(\alpha)) = f \restriction I_\alpha$. If $\xi \in I_\alpha$, then $f(\xi) = \pi_\alpha(f'(\alpha))(\xi) \in \varphi'(\xi)$. Hence $f \in^* \varphi'$.

If D with $|D| = \lambda$ witnesses $\mathfrak{d}^h_\kappa(\in^*) = \lambda$ and $f \in {}^\kappa\kappa$, let f' be as above and $\varphi \in D$ such that $f' \in^* \varphi$. Then $f \in^* \varphi'$, so $\{\varphi' \mid \varphi \in D\}$ witnesses $\mathfrak{d}^g_\kappa(\in^*) \leq \lambda$. $\square$

Contents

Our goal is to separate $\mathfrak{d}^h_\kappa(\in^*)$ from $\mathfrak{d}^g_\kappa(\in^*)$ for two $h, g \in {}^\kappa\kappa$.

Definition

A forcing notion $\langle \mathbb{P}, \leq \rangle$ has the **(generalised)** h**-Sacks property** if for every $\mathbb{P}$-name $\dot{f}$ and condition $p \in \mathbb{P}$ such that $p \Vdash$ "$\dot{f} \in {}^\kappa\kappa$" there exists a $q \leq p$ and h-slalom $\varphi \in \mathrm{Loc}_h$ such that $q \Vdash$ "$\dot{f}(\check{\alpha}) \in \check{\varphi}(\check{\alpha})$" for all $\alpha < \kappa$.

Proposition *see e.g. [Jech, 2003] lemma 15.36 (for $\kappa = \omega$)*

If $\mathbb{P}$ has the h-Sacks property for some $h \in {}^\kappa\kappa$ and $\mathbb{P}$ is $<\kappa$-closed, then $\mathbb{P}$ does not collapse κ^+. □

Lemma

Let $\mathbb{P}$ have the h-Sacks property and preserve cardinals, then $\mathbf{V}[G_\mathbb{P}] \vDash$ "$\mathfrak{d}^h_\kappa(\in^*) \leq (2^\kappa)^\mathbf{V}$". □

Let $T \subseteq {}^{<\kappa}\kappa$ be a tree. For any node $u \in T$ let $\mathrm{suc}(u, T) = \{v \in T \mid \exists \beta < \kappa (v = u^\frown \beta)\}$.

Node u is α-**splitting** in T if $\alpha \leq |\mathrm{suc}(u, T)|$. If u is α-splitting but not $|\alpha|^+$-splitting, then we call u a **sharp** α-splitting node. A **splitting node** is a 2-splitting node, and any other node is **non-splitting**.

We let $u \in \mathrm{Split}_\alpha(T)$ iff u is splitting and $\mathrm{ot}(\{\beta < \mathrm{ot}(u) \mid u \restriction \beta \text{ is splitting}\}) = \alpha$, and we call α the **splitting level** of u.

If $u \in T$, then $T_u = \{v \in T \mid u \subseteq v \text{ or } v \subseteq u\}$.

The **generalised Sacks forcing** or **perfect-set forcing** $\mathbb{S}_\kappa$ has as conditions trees $T \subseteq {}^{<\kappa}2$ such that:

(i) for any $u \in T$ there exists splitting $v \in T$ such that $u \subseteq v$,

(ii) if $\gamma < \kappa$ and $\langle u_\alpha \mid \alpha < \gamma \rangle \in {}^{\gamma}T$ are splitting nodes with $u_\alpha \subseteq u_\beta$ for $\alpha < \beta$, then $u = \bigcup_{\alpha<\gamma} u_\alpha \in T$ and u is splitting.

The ordering on $\mathbb{S}_\kappa$ is given by $T \leq S$ iff $T \subseteq S$.

If $T \in \mathbb{S}_\kappa$, then $\bigcup_{\alpha<\kappa} \mathrm{Split}_\alpha(T)$ is isomorphic with ${}^{<\kappa}2$.

Proposition *[Kanamori, 1980] lemma 1.2*

$\mathbb{S}_\kappa$ is $<\kappa$-closed and has the $<(2^\kappa)^+$-cc. $\square$

This implies that $\mathbb{S}_\kappa$ preserves cardinals $\leq \kappa$ and $<(2^\kappa)$. We will see that $\mathbb{S}_\kappa$ has the pow-Sacks property, and thus preserves κ^+. Hence if $\mathbf{V} \models$ "$2^\kappa = \kappa^+$", then $\mathbb{S}_\kappa$ preserves all cardinals and cofinalities.

Let $T \leq_\alpha S$ iff $T \leq S$ and $\mathrm{Split}_\alpha(T) = \mathrm{Split}_\alpha(S)$. A **fusion sequence** is a sequence $\langle T_\alpha \mid \alpha < \kappa \rangle$ s.t. $T_\beta \leq_\alpha T_\alpha$ for all $\beta > \alpha$.

Proposition *[Kanamori, 1980] lemma 1.4*

$\mathbb{S}_\kappa$ is closed under fusion, that is, if $\langle T_\alpha \mid \alpha < \kappa \rangle$ is a fusion sequence, then there is $S \in \mathbb{S}_\kappa$ such that $S \leq T_\alpha$ for all $\alpha < \kappa$. $\square$

Proposition *[Brendle et al., 2018] proposition 65 & 66*

Let $\mathrm{pow} : \alpha \mapsto 2^\alpha$ and $\mathrm{id} : \alpha \mapsto \alpha$, then $\mathbb{S}_\kappa$ has the pow-Sacks property, but does not have the id-Sacks property.

Proof sketch. If $T \in \mathbb{S}_\kappa$, then $|\mathrm{Split}_\alpha(T)| = 2^{|\alpha|}$. Let $\dot{f}$ be a name such that $T \Vdash$ "$\dot{f} \in {}^\kappa\kappa$". For each $v \in \mathrm{suc}(u, T)$ where $u \in \mathrm{Split}_\alpha(T)$ find an extension of $T'_v \subseteq T_v$ deciding $\dot{f}(\alpha)$ and take the amalgamation of these T'_v. Then use fusion to get increasingly stronger trees $T_{\alpha+1}$ deciding $\dot{f}(\alpha)$.

There are unboundedly many $\alpha < \kappa$ such that $T \cap {}^\alpha 2 = \mathrm{Split}_\alpha(T)$. Let $\dot{f}$ name the $\mathbb{S}_\kappa$-generic κ-real. If $u \in T \cap {}^\alpha 2$, then T_u decides $\dot{f} \restriction \alpha$ and there are $2^{|\alpha|}$ many such u. If φ is an id-slalom, then $|\varphi(\alpha)| = |\alpha| < 2^{|\alpha|}$, thus we can use a bijection $g : \kappa \rightarrowtail\!\!\!\!\twoheadrightarrow 2^{<\kappa}$ to decide a value outside $\varphi(\alpha)$. $\square$

Let $\mathbb{P}$ be a forcing and A be a set of ordinals. For a function $p : A \to \mathbb{P}$, we let $\operatorname{supp}(p) = \{\xi \in A \mid p(\xi) \neq \mathbb{1}_{\mathbb{P}}\}$ be the **support** of p. We define the **$\leq\kappa$-supported A-product of $\mathbb{P}$** as follows:

$$\mathbb{P}^A = \{p : A \to \mathbb{P} \mid |\operatorname{supp}(p)| \leq \kappa\}.$$

If $p, q \in \mathbb{P}^A$, then $q \leq_{\mathbb{P}^A} p$ iff $q(\xi) \leq_{\mathbb{P}} p(\xi)$ for all $\xi \in A$.

Proposition *see e.g. [Jech, 2003] lemma 15.4, 15.12 & 15.17*

If $\mathbb{P}$ is $<\kappa$-closed, then $\mathbb{P}^A$ is $<\kappa$-closed.
If $|\mathbb{P}| \leq \lambda$, then $\mathbb{P}^A$ has the $<\lambda^+$-cc. $\square$

Corollary

$\mathbb{S}_\kappa^A$ is $<\kappa$-closed and has the $<(2^\kappa)^+$-cc.

Given $p, q \in \mathbb{S}^A_\kappa$, $\alpha < \kappa$, and $Z \subseteq A$ with $|Z| < \kappa$, let $q \leq_{Z,\alpha} p$ iff $q \leq p$ and for each $\xi \in Z$ we have $q(\xi) \leq_\alpha p(\xi)$.

A **generalised fusion sequence** is a sequence $\langle (p_\alpha, Z_\alpha) \mid \alpha < \kappa \rangle$ such that:

- $p_\alpha \in \mathbb{S}^A_\kappa$ and $Z_\alpha \in [A]^{<\kappa}$ for each $\alpha < \kappa$,
- $p_\beta \leq_{Z_\alpha,\alpha} p_\alpha$ and $Z_\alpha \subseteq Z_\beta$ for all $\alpha < \beta < \kappa$,
- for limit δ we have $Z_\delta = \bigcup_{\alpha<\delta} Z_\alpha$,
- $\bigcup_{\alpha<\kappa} Z_\alpha = \bigcup_{\alpha<\kappa} \operatorname{supp}(p_\alpha)$.

Proposition *[Kanamori, 1980] lemma 1.9*

$\mathbb{S}^A_\kappa$ is closed under generalised fusion. $\square$

Corollary

If $\mathbf{V} \models$ "$2^\kappa = \kappa^+$", then $\mathbb{S}^A_\kappa$ preserves cardinals and cofinalities. $\square$

Lemma *[Brendle et al., 2018] main lemma 69*

$\mathbb{S}_\kappa^A$ has the pow-Sacks property.

Proof sketch. The proof is the same as before, but for multiple $\mathbb{S}_\kappa$ conditions simultaneously. To construct the fusion sequence $\langle (p_\alpha, Z_\alpha) \mid \alpha < \kappa \rangle$, at stage α we only need to control $p_\alpha(\beta)$ for $\beta \in Z_\alpha$. We can construct the sequence such that $|Z_\alpha| = |\alpha|$ using bookkeeping, hence the amalgamation stays small enough. $\square$

Theorem *[Brendle et al., 2018] theorem 70*

Assume $\mathbf{V} \vDash$ " $2^\kappa = \kappa^+$ ", let $\lambda > \kappa^+$ be regular and let G be $\mathbb{S}_\kappa^\lambda$-generic, then $\mathbf{V}[G] \vDash$ " $\kappa^+ = \mathfrak{d}_\kappa^{\mathrm{pow}}(\in^*) < \mathfrak{d}_\kappa^{\mathrm{id}}(\in^*) = 2^\kappa$ ". $\square$

Contents

In essence, because $\left|\bigcup_{u\in\mathrm{Split}_\alpha(T)}\mathrm{suc}(u,T)\right| = 2^{|\alpha|}$ for $T\in\mathbb{S}_\kappa$, we have enough freedom to make the id-Sacks property fail, but restrict the branching enough to make the pow-Sacks property hold.

Given $h,g\in{}^\kappa\kappa$, let $h\ll g$ denote that $|h(\alpha)|<|g(\alpha)|$ for limit α. Given some $F_0\in{}^\kappa\kappa$, we want to find $F_1\in{}^\kappa\kappa$ such that $F_0\ll F_1$ and a forcing $\mathbb{P}$ such that $\mathbb{P}$ has the F_1-Sacks property, but not the F_0-Sacks property.

Solution: use a tree forcing with perfect trees T, where $u\in\mathrm{Split}_\alpha(T)$ splits more than $F_0(\alpha)$ times, but at most $F_1(\alpha)$ times. We also need $\mathbb{P}$ to preserve cardinals and we need the Sacks properties to be preserved by products or iteration.

Let $h \in {}^\kappa\kappa$ be an increasing cofinal cardinal function. The conditions of the forcing $\mathbb{S}^h_\kappa$ are trees $T \subseteq {}^{<\kappa}\kappa$ that satisfy the following properties:

(i) for any $u \in T$ there exists splitting $v \in T$ such that $u \subseteq v$,

(ii) if $\gamma < \kappa$ and $\langle u_\alpha \mid \alpha < \gamma \rangle \in {}^\gamma T$ are splitting nodes with $u_\alpha \subseteq u_\beta$ for $\alpha < \beta$, then $u = \bigcup_{\alpha<\gamma} u_\alpha \in T$ and u is splitting,

(iii) if $u \in \mathrm{Split}_\alpha(T)$, then u is an $h(\alpha)$-splitting node in T.

We say that $T \leq S$ iff $T \subseteq S$ and for every splitting $u \in T$, either $\mathrm{suc}(u,T) = \mathrm{suc}(u,S)$ or $|\mathrm{suc}(u,T)| < |\mathrm{suc}(u,S)|$.

Proposition

If $T \in \mathbb{S}^h_\kappa$ and $\alpha < \kappa$, then $\left|\bigcup_{u \in \mathrm{Split}_\alpha(T)} \mathrm{suc}(u,T)\right| = h(\alpha)^{|\alpha|}$. $\square$

Proposition *[vdV] lemma 4*

Let $\gamma < \kappa$ and $\langle T_\xi \mid \xi < \gamma \rangle \in {}^\gamma(\mathbb{S}^h_\kappa)$ be decreasing. If $u \in T = \bigcap T_\xi$ is splitting in T_ξ for all $\xi < \lambda$, then u is splitting in T and there is $\eta < \kappa$ such that for all $\xi \in [\eta, \lambda)$ we have $\mathrm{suc}(u, T) = \mathrm{suc}(u, T_\xi)$.

Proof. Let $\lambda_\xi = |\mathrm{suc}(u, T_\xi)|$, then $\langle \lambda_\xi \mid \xi < \gamma \rangle$ is a descending sequence, hence there is $\eta < \gamma$ such that $\lambda_\xi = \lambda_\eta$ for all $\xi \in [\eta, \gamma)$. Thus $\mathrm{suc}(u, T_\xi) = \mathrm{suc}(u, T_\eta)$ for all $\xi \in [\eta, \lambda)$. $\square$

Corollary *[vdV] lemma 4*

$\mathbb{S}^h_\kappa$ is $<\kappa$-closed. $\square$

Proposition *[vdV] lemma 6*

$\mathbb{S}^h_\kappa$ is closed under fusion and has the $<(2^\kappa)^+$-cc. $\square$

For any $T \in \mathbb{S}^h_\kappa$ and $u \in T$, the subtree T_u is a condition.

Every T has a **sharp** $T^* \leq T$ such that $\mathrm{Split}_\alpha(T^*) \subseteq \mathrm{Split}_\alpha(T)$ and each $u \in \mathrm{Split}_\alpha(T^*)$ is a sharp $h(\alpha)$-splitting node.

Theorem *[vdV] theorem 7*

For every $h \in {}^\kappa\kappa$ there exists $F \in {}^\kappa\kappa$ such that $h \leq F$ and $\mathbb{S}^h_\kappa$ has the F-Sacks property. In particular, $F : \alpha \mapsto h(\alpha)^{|\alpha|}$ suffices.

Proof sketch. We use the same idea as pow-Sacks property of $\mathbb{S}_\kappa$.

Let $T_0 \in \mathbb{S}^h_\kappa$ and $\dot{f}$ be a $\mathbb{S}^h_\kappa$-name with $T_0 \Vdash$ "$\dot{f} \in {}^\kappa\kappa$", then we construct a fusion sequence $\langle T_\xi \mid \xi < \kappa\rangle$ and a sequence of sets $\langle A_\xi \mid \xi < \kappa\rangle$ with $|A_\xi| \leq F(\alpha)$ such that $T_{\xi+1} \Vdash$ "$\dot{f}(\check{\xi}) \in \check{A}_\xi$".

We need $u \in \mathrm{Split}_\alpha(T_\xi)$ to have $|\mathrm{suc}(u, T_\xi)| = h(\alpha)$ for each $\alpha < \xi$, hence we make sure T_ξ is sharp for all ξ. *Cont'd*

Cont'd. Given T_ξ, let $V_\xi = \bigcup_{u \in \mathrm{Split}_\xi(T_\xi)} \mathrm{suc}(u, T_\xi)$, then $|V_\xi| \leq h(\xi)^{|\xi|} = F(\xi)$ because T_ξ is sharp. For each $v \in V_\xi$, we find $T^v_\xi \leq (T_\xi)_v$ that decides $\dot{f}(\check{\xi})$. We then fix some successor v' of some $u' \in \mathrm{Split}_\xi(T^v_\xi)$ and let $T'_{\xi+1}$ be the amalgamation of all $(T^v_\xi)_{v'}$ with $v \in V_\xi$. Finally we let $T_{\xi+1} = (T'_{\xi+1})^*$ be sharp.

A_ξ consists of the values that each T^v_ξ decided for $\dot{f}(\check{\xi})$.

For limit γ we take $T_\gamma = (\bigcap_{\xi<\gamma} T_\xi)^*$. $\square$

Corollary

$\mathbb{S}^h_\kappa$ preserves κ^+.

Corollary

If $\mathbf{V} \vDash$ "$2^\kappa = \kappa^+$", then $\mathbb{S}^h_\kappa$ preserves all cardinals and cofinalities.

Theorem *[vdV] theorem 9*

Let $F, h \in {}^\kappa\kappa$ and $F \ll h$, then $\mathbb{S}^h_\kappa$ does not have the F-Sacks property.

Proof sketch. Similar to the failure of id-Sacks property for $\mathbb{S}_\kappa$.

Let φ be an F-slalom, $T \in \mathbb{S}^h_\kappa$, and $\dot{f}$ name the $\mathbb{S}^h_\kappa$-generic κ-real.

There are unboundedly many limit $\alpha < \kappa$ s.t. $T \cap {}^\alpha\kappa = \mathrm{Split}_\alpha(T)$. If $u \in T \cap {}^{\alpha+1}\kappa$, then T_u decides $\dot{f}(\alpha)$ and there are $h(\alpha)^{|\alpha|}$ many such u. Since $|\varphi(\alpha)| = F(\alpha) < h(\alpha)$, we can choose u with $u(\alpha) \notin \varphi(\alpha)$ to see that $T_u \Vdash$ "$\dot{f}(\check{\alpha}) \notin \check{\varphi}(\check{\alpha})$". By denseness it follows that $\Vdash$ "$\dot{f} \notin^* \check{\varphi}$". $\square$

Lemma *[vdV] lemma 10, 11, 12*

Let A be a set of ordinals, then $(\mathbb{S}^h_\kappa)^A$ is $<\kappa$-closed, has the $<(2^\kappa)^+$-cc and is closed under generalised fusion. $\square$

Lemma *[vdV] lemma 13*

If $\mathbb{S}^h_\kappa$ has the F-Sacks property, then $(\mathbb{S}^h_\kappa)^A$ has the F-Sacks property. $\square$

Theorem *[vdV] theorem 14*

Let $F_0, h \in {}^\kappa\kappa$ be increasing cofinal cardinal functions such that $F_0 \ll h$ and let $F_1 : \alpha \mapsto h(\alpha)^{|\alpha|}$. Assuming that $\mathbf{V} \vDash$ "$2^\kappa = \kappa^+$" and $\lambda > \kappa^+$ is regular, for any $(\mathbb{S}^h_\kappa)^\lambda$-generic G we have $\mathbf{V}[G] \vDash$ "$\mathfrak{d}^{F_1}_\kappa(\in^*) = \kappa^+ < \mathfrak{d}^{F_0}_\kappa(\in^*) = \lambda = 2^\kappa$".

Proof sketch. First, $\mathfrak{d}^{F_1}_\kappa(\in^*) = \kappa^+$ by the F_1-Sacks property.

That $\lambda = 2^\kappa$ is a standard argument.

Working in $\mathbf{V}[G]$, let $\kappa^+ \leq \mu < \lambda$ and suppose that $D = \{\varphi_\xi \mid \xi < \mu\} \subseteq \mathrm{Loc}_{F_0}$ witnesses that $\mathfrak{d}^{F_0}_\kappa(\in^*) = \mu < \lambda$. Since F_0-slaloms are essentially κ-reals, it follows that there is $A \subseteq \lambda$ with $|A| \leq \mu$ such that $D \in \mathbf{V}[G \restriction A]$.

We may then pick $\beta \in \lambda \setminus A$ and let f be the $\mathbb{S}^h_\kappa$-generic κ-real added in the β-th term of the product. The proof that $\mathbb{S}^h_\kappa$ does not have the F_0-Sacks property, implies that $f \notin^* \varphi_\xi$ for any $\xi < \mu$. $\square$

Contents

Open problems

- Can we separate multiple $\mathfrak{d}_\kappa^h(\in^*)$ simultaneously?
  - Let $\kappa^+ < \lambda < \mu$, and $\mathrm{id} \lll h \lll h'$, use $(\mathbb{S}_\kappa^h)^\mu \times (\mathbb{S}_\kappa^{h'})^\lambda$
- If $h \lll h'$, does there exists $h \leq g \leq h'$ such that $\mathfrak{d}_\kappa^g(\in^*)$ is consistently different from $\mathfrak{d}_\kappa^h(\in^*)$ and $\mathfrak{d}_\kappa^{h'}(\in^*)$?
- If are there h and h' such that both $\mathfrak{d}_\kappa^h(\in^*) < \mathfrak{d}_\kappa^{h'}(\in^*)$ and $\mathfrak{d}_\kappa^{h'}(\in^*) < \mathfrak{d}_\kappa^h(\in^*)$ are consistent?
  - We need stationary sets S, S' such that $h(\alpha) \leq h'(\alpha)$ for all $\alpha \in S$ and $h'(\alpha) \leq h(\alpha)$ for all $\alpha \in S'$.
- Can we separate $\mathfrak{b}_\kappa^h(\in^*)$ for different functions $h \in {}^\kappa\kappa$?
  - We cannot dualise the forcing, as we need $\mathbf{V} \vDash 2^\kappa = \kappa^+$.
- What is the relation between the slalom cardinals and Shelah's "null ideal" for inaccessible λ?
  - Partial results: [Baumhauer et al., 2020]

References

Tomek Bartoszyński. Combinatorial aspects of measure and category. **Fundamenta Mathematicae**, 127(3):225–239, 1987.

Tomek Bartoszyński and Haim Judah. **Set Theory: On the Structure of the Real Line**. A.K. Peters, Wellesley, MA, 1995.

Thomas Baumhauer, Martin Goldstern, and Saharon Shelah. The Higher Cichoń Diagram. **Fundamenta Mathematicae**, 252(3):241–314, 2020.

Andreas Blass. Combinatorial Cardinal Characteristics of the Continuum. In **Handbook of Set Theory Vol. 1**, pages 395–489. Springer, Dordrecht, 2010.

Jörg Brendle, Andrew Brooke-Taylor, Sy-David Friedman, and Diana Carolina Montoya. Cichoń's diagram for uncountable cardinals. **Israel Journal of Mathematics**, 225(2):959–1010, 2018.

Thomas Jech. **Set Theory: Third Millennium Edition**. Springer Monographs in Mathematics, 2003.

Akihiro Kanamori. Perfect-set forcing for uncountable cardinals. **Annals of Mathematical Logic**, 19(1-2):97–114, 1980.

Tristan van der Vlugt. Separating Many Slalom Cardinals. Unpublished notes: `http://tvdvlugt.nl/separatingslaloms.pdf`, 2021.