Determinacy and Supercompactness of $\aleph_1$

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Strongly Compact and Supercompact cardinals

Recall that $\mathcal{P}_\kappa(A) := \{a \subseteq A : a \text{ injects into } \kappa \text{ and } |a| < \kappa\}$. An ultrafilter $\mathcal{U}$ on $\mathcal{P}_\kappa(A)$ is

▶ fine if $\{a \in \mathcal{P}_\kappa(A) : x \in a\} \in \mathcal{U}$ for all $x \in A$.

▶ normal if for every collection $\langle A_x : x \in A \rangle$ with $A_x \in \mathcal{U}$

$$\triangle_{x \in A} A_x := \{a \in \mathcal{P}_\kappa(A) : a \in \bigcap_{x \in a} A_x\} \in \mathcal{U}.$$
Strongly Compact and Supercompact cardinals

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- **fine** if $\{ a \in \mathcal{P}_\kappa(A) : x \in a \} \in U$ for all $x \in A$.
- **normal** if for every collection $\langle A_x : x \in A \rangle$ with $A_x \in U$

$$\bigtriangleup_{x \in A} A_x := \{ a \in \mathcal{P}_\kappa(A) : a \subseteq \bigcap_{x \in a} A_x \} \in U.$$

**Definition**

Let $\kappa$ be a cardinal and $A$ a set. We say $\kappa$ is

- **$A$-strongly compact** if there is a fine, $\kappa$-complete ultrafilter on $\mathcal{P}_\kappa(A)$.
- **$A$-supercompact** if there is a fine, normal, $\kappa$-complete ultrafilter on $\mathcal{P}_\kappa(A)$. 
Suslin cardinals

Recall $\theta = \sup\{\nu : \mathbb{R} \text{ surjects onto } \nu\}$.

A set $X \subseteq \mathbb{R}$ is $\lambda$-Suslin if there is some tree $T$ on $\omega \times \lambda$ such that

$$X = p[T] := \{x \in \mathbb{R} : T_x \text{ is ill-founded}\}$$

**Definition**

A cardinal $\lambda$ is a *Suslin cardinal* if there is a $\lambda$-Suslin set, that is not $\gamma$-Suslin for any $\gamma < \lambda$.

Note that any Suslin cardinal is less than $\theta$. 
$\mathbb{R}$-supercompactness of $\aleph_1$ under $\text{AD}_\mathbb{R}$

For $A \subseteq \mathcal{P}_{\omega_1}(\mathbb{R})$ consider the game

\[
\begin{array}{cccccc}
I & & a_0 & a_2 & a_4 & \ldots & a_i \in \mathcal{P}_\omega(\mathbb{R}) \\
\| & a_1 & a_3 & a_5 & \quad & \quad &
\end{array}
\]

where player II wins if $\bigcup_{i<\omega} a_i \in A$. 
$\mathbb{R}$-supercompactness of $\aleph_1$ under $\text{AD}_\mathbb{R}$

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II & \quad a_1 \quad a_3 \quad a_5
\end{align*}

where player II wins if $\bigcup_{i<\omega} a_i \in A$.

$\mathcal{U} = \{ A \subseteq \mathcal{P}_{\omega_1}(\mathbb{R}) : \text{player II has a winning strategy in this game} \}$
R-supercompactness of $\aleph_1$ under $\text{AD}_R$

For $A \subseteq P_{\omega_1}(\mathbb{R})$ consider the game

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Theorem (Solovay, 1978)

$(\text{AD}_R)$ $\mathcal{U}$ is a normal measure. Hence $\aleph_1$ is $<\theta$-supercompact.
Question: How much supercompactness of $\aleph_1$ do we get from various weakenings of $AD_{\mathbb{R}}$?
The Harrington-Kechris result

Theorem (Harrington, Kechris, 1981)

(AD) Suppose $\lambda$ is below a Suslin cardinal, then $\aleph_1$ is $\lambda$-supercompact.
AD$^+$ and supercompactness of $\aleph_1$

By $\lambda$-determinacy we mean the assertion that for any continuous function $f : \lambda^\omega \to \mathbb{R}$ and any $A \subseteq \mathbb{R}$ the game $G(f^{-1}(A))$ played on $\lambda$ with payoff set $f^{-1}(A)$ is determined.
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**Definition**

AD$^+$ is the conjunction of the following:

- $\text{DC}_\mathbb{R}$
- $\lambda$-determinacy for $\lambda < \theta$
- every set of reals is $\infty$-Borel

Note that AD$^+ \rightarrow$ AD and AD$^+ \Rightarrow$ DC$\mathbb{R}$, however AD$^+ \rightarrow$ AD$^+$ is still open.
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Note that AD$^+ \rightarrow$ AD and AD$^\mathbb{R} +$ DC $\rightarrow$ AD$^+$. However

$$AD^\mathbb{R} \rightarrow AD^+$$

is still open.
The reason we are interested in $\text{AD}^+$ is that

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**Theorem**

$(\text{AD}^+)$ Suppose $\lambda$ is a Suslin cardinal, then $\aleph_1$ is $\lambda$-supercompact.
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The reason we are interested in AD$^+$ is that

$$L(\mathbb{R}) \models \text{AD} \rightarrow \text{AD}^+. $$

**Theorem**

(AD$^+$) Suppose $\lambda$ is a Suslin cardinal, then $\aleph_1$ is $\lambda$-supercompact.

So assuming AD, $\aleph_1$ is $\lambda$-supercompact in $L(\mathbb{R})$. 
The supercompact measure on $\aleph_1$

Let $\lambda$ be an ordinal. For $A \subseteq \mathcal{P}_{\omega_1}(\lambda)$ consider the game

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where player II wins if $\bigcup_{i<\omega} a_i \in A$. 

$U = \{ A \subseteq \mathcal{P}_{\omega_1}(\lambda) : \text{player II has a winning strategy in this game} \}$

Under ZF this $U$ is always a filter. The AD$^+$ proof shows this is a normal measure on $\aleph_1$ (for $\lambda$ a Suslin cardinal).
The supercompact measure on $\mathfrak{N}_1$

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- The AD$^+$ proof shows this is a normal measure on $\aleph_1$ (for $\lambda$ a Suslin cardinal)
The filter $\mathcal{U}$

Proposition

$(\text{ZF} + \text{DC})$ Every set in $\mathcal{U}$ contains a club set.
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$\quad\quad\quad$ $\implies$ $(DC + AD^+)$ For $\lambda$ a Suslin cardinal the club (ultra-)filter on $\mathcal{P}_{\omega_1}(\lambda)$ is a supercompact measure on $\aleph_1$
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**Theorem (Woodin, 1983)**

$(\text{ZF} + \text{DC})$ If $\mathcal{V}$ is a supercompact measure on $\aleph_1$ then $\mathcal{U} \subseteq \mathcal{V}$.
The filter $\mathcal{U}$

Proposition
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$\Rightarrow$ $(DC + AD^+)$ For $\lambda$ a Suslin cardinal the club (ultra-)filter on $\mathcal{P}_{\omega_1}(\lambda)$ is a supercompact measure on $\aleph_1$

Theorem (Woodin, 1983)
$(ZF + DC)$ If $\mathcal{V}$ is a supercompact measure on $\aleph_1$ then $\mathcal{U} \subseteq \mathcal{V}$.

Corollary
$(DC + AD^+)$ If $\lambda$ is a Suslin cardinal, $\aleph_1$ is $\lambda$-supercompact and the club filter is the unique $\lambda$-supercompact measure on $\aleph_1$. 
AD and Strong Compactness of $\aleph_1$

Theorem

(AD) If $\lambda < \theta$ then $\aleph_1$ is $\lambda$-strongly compact.