Determinacy and Supercompactness of $\aleph_1$

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16 April 2020

Strongly Compact and Supercompact cardinals

Recall that $\mathcal{P}_\kappa(A) := \{a \subseteq A : a \text{ injects into } \kappa \text{ and } |a| < \kappa\}$. An ultrafilter $\mathcal{U}$ on $\mathcal{P}_\kappa(A)$ is

- *fine* if $\{a \in \mathcal{P}_\kappa(A) : x \in a\} \in \mathcal{U}$ for all $x \in A$.
- *normal* if for every collection $\langle A_x : x \in A\rangle$ with $A_x \in \mathcal{U}$

$$\triangle_{x \in A} A_x := \{a \in \mathcal{P}_\kappa(A) : a \in \bigcap_{x \in a} A_x\} \in \mathcal{U}.$$

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Definition

Let κ be a cardinal and A a set. We say κ is

- *A-strongly compact* if there is a fine, κ-complete ultrafilter on $\mathcal{P}_\kappa(A)$.
- *A-supercompact* if there is a fine, normal, κ-complete ultrafilter on $\mathcal{P}_\kappa(A)$.

Suslin cardinals

Recall $\theta = \sup\{\nu : \mathbb{R} \text{ surjects onto } \nu\}$.

A set $X \subseteq \mathbb{R}$ is λ-Suslin if there is some tree T on $\omega \times \lambda$ such that

$$X = p[T] := \{x \in \mathbb{R} : T_x \text{ is ill-founded}\}$$

Definition

A cardinal λ is a *Suslin cardinal* if there is a λ-Suslin set, that is not γ-Suslin for any $\gamma < \lambda$.

Note that any Suslin cardinal is less than θ.

$\mathbb{R}$-supercompactness of $\aleph_1$ under $\mathsf{AD}_{\mathbb{R}}$

For $A \subseteq \mathcal{P}_{\omega_1}(\mathbb{R})$ consider the game

$$\begin{array}{lcccccccc} \text{I} & a_0 & & a_2 & & a_4 & & \ldots & a_i \in \mathcal{P}_\omega(\mathbb{R}) \\ \text{II} & & a_1 & & a_3 & & a_5 & & \end{array}$$

where player II wins if $\bigcup_{i<\omega} a_i \in A$.

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Theorem (Solovay, 1978)

($AD_{\mathbb{R}}$) *This $\mathcal{U}$ is a normal measure. Hence $\aleph_1$ is $<\theta$-supercompact.*

Question: How much supercompactness of $\aleph_1$ do we get from various weakenings of $\mathsf{AD}_{\mathbb{R}}$?

The Harrington-Kechris result

Theorem (Harrington, Kechris, 1981)

(AD) *Suppose* λ *is below a Suslin cardinal, then* $\aleph_1$ *is* λ*-supercompact.*

AD^+ and supercompactness of $\aleph_1$

By λ-determinacy we mean the assertion that for any continuous function $f : \lambda^\omega \to \mathbb{R}$ and any $A \subseteq \mathbb{R}$ the game $G(f^{-1}(A))$ played on λ with payoff set $f^{-1}(A)$ is determined.

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Definition

AD^+ is the conjunction of the following:

- $DC_{\mathbb{R}}$
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- every set of reals is ∞-Borel

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Note that $\mathsf{AD}^+ \to \mathsf{AD}$ and $\mathsf{AD}_\mathbb{R} + \mathsf{DC} \to \mathsf{AD}^+$. However

$$\mathsf{AD}_\mathbb{R} \to \mathsf{AD}^+$$

is still open.

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So assuming AD, $\aleph_1$ is λ-supercompact in $L(\mathbb{R})$.

The supercompact measure on $\aleph_1$

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- Under ZF this $\mathcal{U}$ is always a filter
- The AD^+ proof shows this is a normal measure on $\aleph_1$ (for λ a Suslin cardinal)

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Theorem (Woodin, 1983)

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Corollary

($\mathsf{DC} + \mathsf{AD}^+$) *If λ is a Suslin cardinal, $\aleph_1$ is λ-supercompact and the club filter is the unique λ-supercompact measure on $\aleph_1$.*

AD and Strong Compactness of $\aleph_1$

Theorem

(AD) *If* $\lambda < \theta$ *then* $\aleph_1$ *is* λ*-strongly compact.*