Inner models of measurability

κ measurable :<=> ex κ-complete normal Ultrafilter over κ.

investigate $\rightsquigarrow$ investigate $L[U]$

constructible universe relative to U

smallest inner model M with $x \cap U \in M$ for all $x \in M$.

<u>Def.</u> $L_0[U] = \emptyset$

$$L_{\alpha+1}[U] = \mathrm{def}^U(L_\alpha[U])$$

$$\mathrm{def}^U(x) := \{ y \subseteq x \mid y \text{ definable over } (x, \in, U \cap x) \}$$

$$L_\delta[U] = \bigcup_{\alpha<\delta} L_\alpha[U]$$

$$L[U] := \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha[U]$$

<u>Fact</u> $L[\mathbb{R}] = L$

- Iterated Ultrapowers
- κ measurable $\Rightarrow$ $({}^{\kappa}M/U, \in_U)$ well-founded

<u>Def</u> $(M, \in, U)$ iterable if the UP iteration ranges over Ord.

<u>Fact</u> $(L[U], \in, U)$ is iterable, i.e. for all limits δ $\operatorname{dir\,lim}_{\alpha<\delta} (M_\alpha)$ well-founded.

U measure over κ

<u>Def</u> $(M, \in, U)$ and $(N, \in, W)$ are comparable if

$U = W \cap M$ or $W = U \cap N$.

<u>Fact</u> $(L[U], \in, U)$ and $(L[W], \in, W)$ always have comparable iterates.

Thm Let $(L[U], \in, U)$, $(L[W], \in, W)$ be κ-models, then $U = W$ and $L[U] = L[W]$.

Def. $(L[U], \in, U)$ is κ-model if $\left(U \in L[U] \right.$ $(L[U], \in, U) \models$ "U is a normal UF over κ" $\left. \right)$

Beweis: By fact $(L[U], \in, U)$ and $(L[W], \in, W)$ have a common iterator $(L[F], \in, F)$ of both.

$$i : (L[U], \in, U) \prec (L[F], \in, F)$$

$$j : (L[W], \in, W) \prec (L[F], \in, F)$$

Then $\{ \theta > \kappa \mid \theta = |\theta| = i(\theta) = j(\theta) \}$

is a proper class.

Let B be any subset of this class with $|B| \geq \kappa^+$.
Let δ be ordinal $> \sup(B)$.

It holds: for $X \in U$ there are definable term t and $\eta_1, \ldots, \eta_n \in B$ with $\xi_1, \ldots, \xi_m \in \kappa$

$$X = t^{(L_\delta[U], \in, U)}(\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n) \qquad (*)$$

Set

$$Y := t^{(L_\delta[W], \in, W)}(\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n) \in W$$

$$i(X) = i\left(t^{(L_\delta[W], \in, W)}(\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n)\right) = t^{(L_\delta[F], \in, F)}(\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n) = j(Y)$$

For every $X \in F$ $\quad i(X) = j(Y) \in F$.

$X = i(X) \cap \kappa = j(Y) \cap \kappa \Rightarrow X \in W$

So $U \subseteq W$ and $W \subseteq U$ analogously.

$\square$

<u>Cor.</u> $(L[U], \in, U)$ κ-model $\Rightarrow$ U is unique normal UF over κ in $L[U]$.

<u>Proof:</u> Let W be normal UF over κ in $L[U]$.

Let $\overline{W} := W \cap L[W]$.

$L[\overline{W}]$ is κ-model

$\Rightarrow \overline{W} = U$

Therefore $U \subseteq W$ and $U = W$ since both UF.

$\square$

<u>Remark</u> $(L[A], \in, A)$ κ-model $\Rightarrow A \in L[A]$.

Lemma for (*) $(L[U], \in, U)$ κ-model, $B \subseteq \mathrm{Ord}$, $|B| \geq \kappa^+$

δ limit ordinal with $B \subseteq L_\delta[U]$.

Then $\mathcal{P}(\kappa) \cap L[U]$ is a subset of Skolem hull of $\kappa \cup B$ in $(L_\delta[U], \in, U)$, i.e.

for any $X \in \mathcal{P}(\kappa) \cap L[U]$

$$X = t^{(L_\delta[U], \in, U)}(\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_n)$$

$\xi_1, \dots, \xi_m \in \kappa$, $\eta_1, \dots, \eta_n \in B$, t definierbarer Term.

<u>Proof</u> Let $(H, \in, U \cap H)$ with H Skolem hull of $\kappa \cup B$. Let $(N, \in, W)$ be its transitive collapse, with map π.

For every $\alpha < \kappa$ holds

$\pi(\alpha) = \alpha$

Therefore $\pi(X) = X$ for $X \in H \cap \mathcal{P}(\kappa)$.

$W = U \cap N$ and

$N = L_\xi[U]$ with $\xi \geq \kappa^+$ since $|B| \geq \kappa^+$

This implies $\mathcal{P}(\kappa) \cap L[U] \subseteq N$

and $\mathcal{P}(\kappa) \cap L[U] \subseteq H$ since $\pi(X) = X$ for $X \in \mathcal{P}(\kappa)$

$\square$

Thm $(L[U], \in, U)$ κ-model

$(L[W], \in, W)$ λ-model

and $\kappa < \lambda$.

Then $(L[W], \in, W)$ is an iterate of $(L[U], \in, U)$.

In particular $L[W] \subseteq L[U]$

and $L[W]$ definable in $(L[U], \in, U)$

with parameter λ.