Automorphisms of $P(\omega)/fin$.

Want to understand automorphisms of infinite structures.

First structure: $(\omega, =)$.

Automorphism group: $S(\omega)$.

$S(\omega)$ has a maximal normal subgroup,

$fin = \{\sigma \in S(\omega) : \sigma \text{ moves only finitely many points}\}$

$S(\omega)/_{fin}$ is an uncountable simple group. (no nontriv. normal subgroup)

$S(\omega)$ is also the group of automorphisms of the Boolean algebra $\mathcal{P}(\omega)$.

Stone duality: $\mathrm{Ult}(\mathcal{P}(\omega)) = \{p : p \subseteq \mathcal{P}(\omega)$ is an ultrafilter$\}$

is topologized by declaring sets of the following form to be open:

For $A \subseteq \omega$ let $\hat{A} = \{p \in \mathrm{Ult}(P(\omega)) : A \in p\}$.

$\mathrm{Ult}(P(\omega))$ is a compact 0-dimensional space.

Identify $n \in \omega$ with the UF gen. by $\{n\}$.

Write $\beta\omega$ for $\mathrm{Ult}(P(\omega))$.

$\omega \subseteq \beta\omega$. $\beta\omega$ is the Stone-Čech compactification of ω.

I.e., $\omega \subseteq \beta\omega$ is dense and for all compact X and $f: \omega \to X$ there is an extension $\bar{f}: \beta\omega \to X$ that is continuous.

$|\beta\omega| = 2^{2^{\aleph_0}}$ There are no nontrivial convergent sequences in $\beta\omega$.

$\mathrm{Hom}(\beta\omega) = \mathrm{Aut}(P(\omega))$.

$\omega^* = \beta\omega \setminus \omega$ This is the Stone-Čech-remainder.

Natural question: Is ω^* homogeneous?

(X is homogeneous $\Leftrightarrow$ for all $x, y \in X$ ex. $h \in \mathrm{Hom}(X)$ s.t. $h(x) = y$)

Stone Duality revisited:

If $p \in \mathrm{Ult}(P(\omega))$ contains a finite set, then p is principal.

If p contains no finite set, then it is closed under finite modifications.

Now consider $fin = \{A \subseteq \omega : A \text{ is finite}\}$.

$$P(\omega)/_{fin} = \{[A]_{=^*} : A \subseteq \omega\}, \quad A =^* B \iff |A \triangle B| < \aleph_0.$$

Nonprincipal ultrafilters in $P(\omega)$ give rise to ultrafilters in $P(\omega)/_{fin}$.

$$\omega^* = \mathrm{Ult}(P(\omega)/_{fin})$$

Theorem (Rudin) Assume CH.
Then there are $x, y \in \omega^*$ s.t. no $h \in \mathrm{Hom}(\omega^*)$ map x to y.

Def: Let X be a top. space and $p \in X$.
p is a P-point if the intersection of only countably many neighborhoods is a neighborhood.

Lemma: CH $\Rightarrow$ ω^* has a P-point.

Proof: Let $(A_\alpha)_{\alpha<\omega_1}$ be an enumeration of all subsets of ω.
Construct sequence $(B_\alpha)_{\alpha<\omega_1}$ s.t.
$B_0 \supseteq^* B_1 \supseteq^* \dots \supseteq^* B_\omega \supseteq^* \dots$, all B_α infinite subsets of ω.

$A_{-1}=\omega$ For all $\alpha<\omega_1$, we want that either
$B_{\alpha+1} \subseteq^* A_\alpha$ or $B_{\alpha+1} \subseteq^* \omega\setminus A_\alpha$.

This can be done:

Successor step: A_α, B_α are given.

Let $B_{\alpha+1} = A_\alpha \cap B_\alpha$ or $B_\alpha \setminus A_\alpha$, depending on which is infinite.

Limit step: α a limit ordinal. Find $B_\alpha \subseteq^* B_\beta$, $\beta<\alpha$.

Let p be the UF generated by $\{B_\alpha : \alpha<\omega_1\}$.

Now let U_n, $n\in\omega$, be nbhds of p in $\beta\omega$.

Wlog. $U_n = \widehat{B_{\alpha_n}}$. Let $\beta = \sup_n \alpha_n$.

$\widehat{B_\beta} \subseteq \widehat{B_{\alpha_n}}$ for all n. $\square$

Lemma: If X is compact and all $x \in X$ are P-points, then X is finite.

Proof: Suppose X is infinite.

Let $(x_n)_{n \in \omega}$ be a sequence w/o repetition in X.

Not needed { We may assume that the sequence does not converge to one of its elements.

Let $y \in X$. For $n \in \omega$ let $U_n = X \setminus \{x_n\}$.

If $y \notin \{x_n : n \in \omega\}$, then each U_n is a neighborhood of y. $\bigcap U_n$ is a nbhd of y

$\Rightarrow$ y is not an accumulation point of the sequence.

Same argument works for the x_n. $\Rightarrow$ Sequence has no acc. points $\lightning$

This finishes the proof of Rudin's thm.

What if we drop CH?

Shelah: It is consistent that there are no P-points in ω^*.

Frolik: ω^* is not homogeneous.

Kunen: ω^* containing a weak p-point (in ZFC).

x is a weak P-point if it is not in the closure of any countable set that does not contain x.

How to construct automorphisms of $P(\omega)/fin$?

Let $A, B \subseteq \omega$ be s.t. $\omega \setminus A$, $\omega \setminus B$ are finite.

Let $b: A \to B$ be a bijection.

b induces an automorphism of $P(\omega)/fin$.

These are called trivial.

How many trivial aut's are there? $2^{\aleph_0}$.

$\boxed{\text{Rudin: } CH \Rightarrow |Aut(P(\omega)/fin)| = 2^{2^{\aleph_0}}}$.

We know: $CH \Rightarrow Aut(P(\omega)/fin)$ is simple.

Shelah: It is consistent that there are no nontrivial autom.

van Douwen: The group of trivial automorphisms is not simple.

Sketch of proof of CH $\Rightarrow$ there are many autom.

Suppose $A, B \subseteq P(\omega)/fin$ are countable BA's

and $f: A \to B$ is an isomorphism.

Let $a \in P(\omega)/fin$.

We want to extend f to an isomorphism

$\bar{f}: A(a) \longrightarrow B(b)$ for some suitable b.

$A \uparrow a = \{x \in A : a \leq x\}$, $A \restriction a = \{x \in A : x \leq a\}$

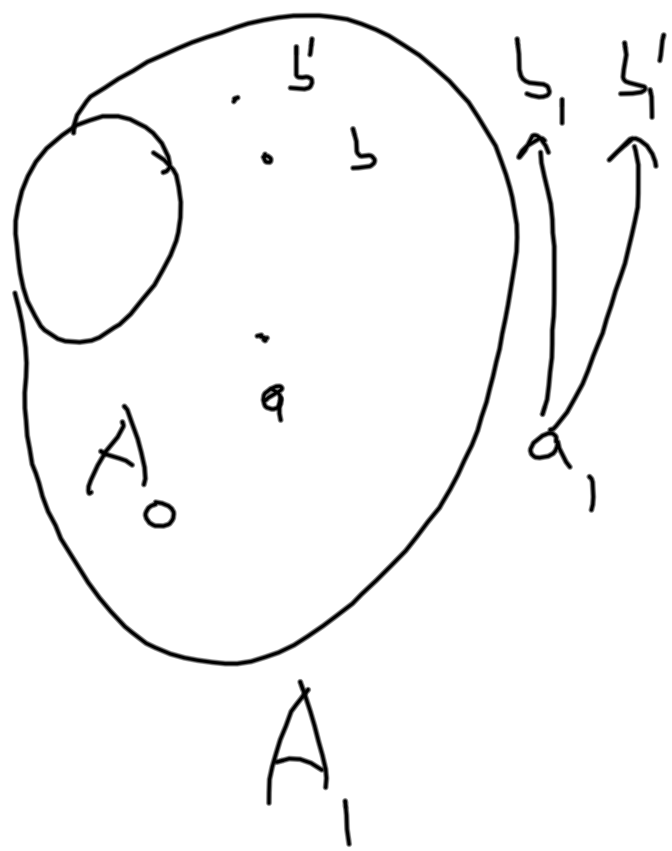
$A \| a = \{x \in A : x \not\leq a, a \not\leq x\}$.

Task: Find $b \in P(\omega)/fin$ s.t.

$$f[A \uparrow a] = B \uparrow b$$
$$f[A \restriction a] = B \restriction b,$$
$$f[A \| a] = B \| b.$$

In $P(\omega)/\mathrm{fin}$, a suitable b exists.
We can even find $b \neq b'$ that both do the job.

Start with $A_0 = \{0,1\} \subseteq P(\omega)/\mathrm{fin}$



$\square$

$\mathcal{A} \subseteq P(\omega)$ is independent

$\Longleftrightarrow$ for all finite, disjoint set $A, B \subseteq \mathcal{A}$

$\bigcap A \cap (\omega \setminus \bigcup B)$ is infinite.

Exist independent F. of size $2^{\aleph_0}$.

$S: \omega \to \omega;\ n \mapsto n+1$

Induces $s \in \mathrm{Aut}(P(\omega)/\mathrm{fin})$

Question: Is $(P(\omega)/\mathrm{fin}, s) \cong (P(\omega)/\mathrm{fin}, s^{-1})$

$$\varphi s \varphi^{-1} = s^{-1}$$