Tower number $\mathfrak{t}$

Def Assume A, B are sets. Define

$$A =^* B \;:\Longleftrightarrow\; A \setminus B \text{ and } B \setminus A \text{ are finite}$$

$$A \subseteq^* B \;:\Longleftrightarrow\; A \setminus B \text{ is finite}$$

Remark $(\mathcal{P}(\omega), \subseteq^*)$ is a preorder

$(\mathcal{P}(\omega)/{=^*}, \subseteq^*)$ is a partial order.

Def Let $\mathcal{F}$ be a family of sets.

A set A is <u>pseudo-intersection</u> of $\mathcal{F}$

iff $A \subseteq^* F$ for all $F \in \mathcal{F}$, and $|A| \geq \omega$.

Def $(T_\alpha \mid \alpha < \lambda)$ is called a tower if

(1) $T_\alpha \in [\omega]^\omega$ for all $\alpha < \lambda$,

(2) $T_\beta \subseteq^* T_\alpha$ for all $\alpha < \beta < \lambda$,

(3) $\{T_\alpha \mid \alpha < \lambda\}$ does NOT have any pseudo-insection.

Set

$$\mathfrak{t} := \min \left\{ \lambda \in \mathrm{Ord} \;\middle|\; \begin{array}{l} \text{there is a tower} \\ (T_\alpha \mid \alpha < \lambda) \text{ of length } \lambda \end{array} \right\}$$

and call it the <u>tower number</u>.

Prop $\underline{t}$ is a regular uncountable cardinal.

Proof: $\underline{t}$ is not singular, since cofinal subsequence of a tower is a tower as well.

$\underline{t} = cf(\underline{t})$

Let $(T_n \mid n<\omega)$ be a sequence fullfiling (1) and (2).

Show that it fails on (3).

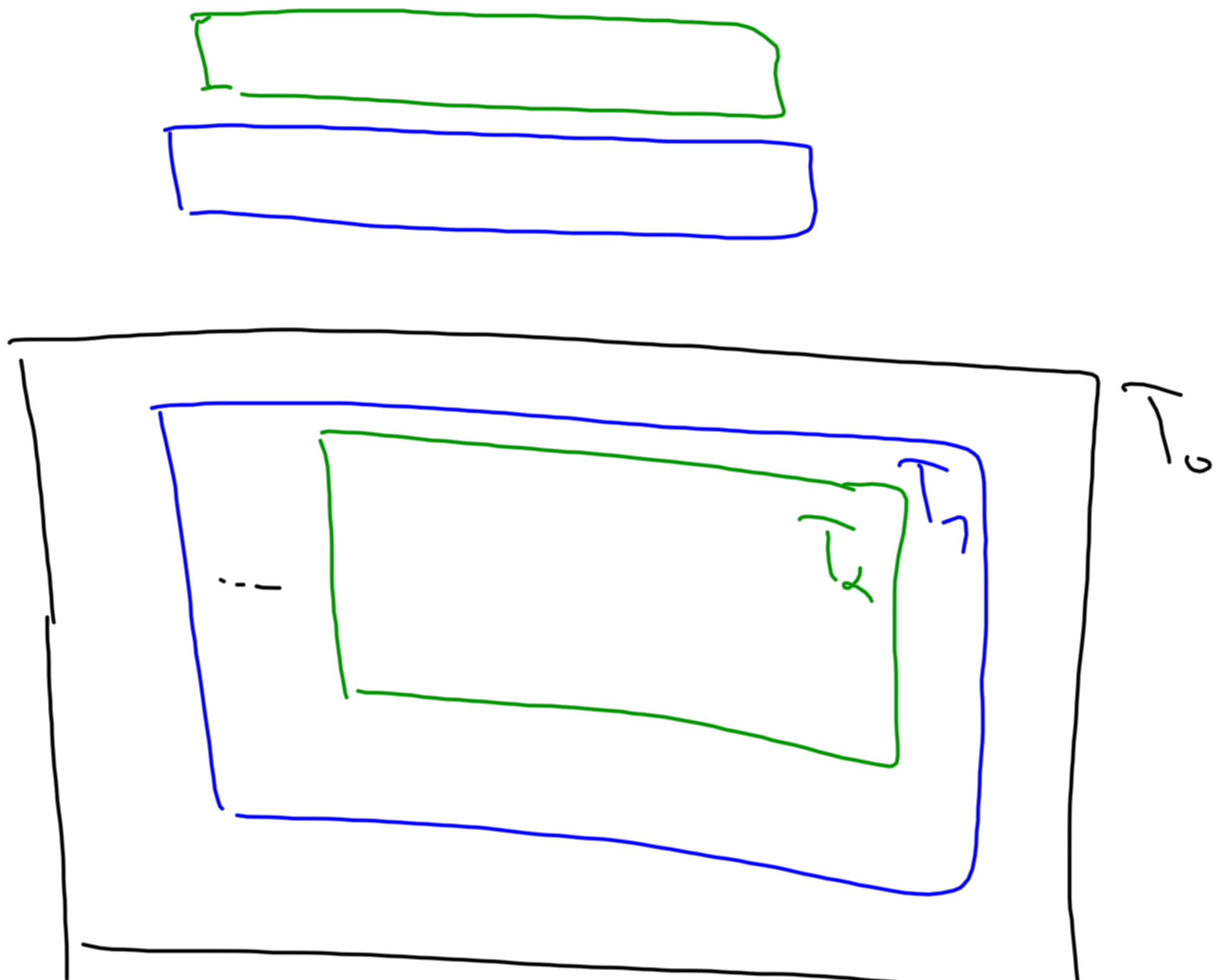
Define $x_0 \in T_0$.

If x_m for $m<n$ is defined, then

take $x_n \in \bigcap_{m<n} T_m$ (is inf.) $x_n \neq x_m$ for $m<n$.

$$\{x_n \mid n<\omega\} \subseteq^* T_n \text{ for all } n<\omega.$$

□



T_0
T_1
T_2
...

Fact $MA(\kappa) \Rightarrow 2^\kappa = 2^\omega$.

Thm If $\omega \leq \kappa < \mathfrak{t}$ then $2^\kappa = 2^\omega$.

Proof: $2^\omega \leq 2^\kappa$ is clear.

Show $2^\kappa \leq 2^\omega$.

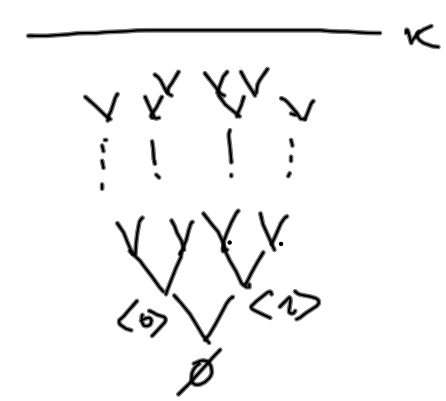
We construct $(T_s \mid s \in {}^{\leq\kappa}2)$ along the full binary tree ${}^\kappa 2$ of height $\kappa+1$ with the properties:

(1) $T_s \in [\omega]^\omega$ for all $s \in {}^{\leq\kappa}2$,

(2) If s_1 initial segment of s_2 then $T_{s_2} \subseteq^* T_{s_1}$

(3) If neither s_1 extends s_2 nor s_2 extends s_1 $T_{s_1} \cap T_{s_2}$ is finite.

Given a sequence with the above properties then



$s_1, s_2 \in {}^{\kappa}2$ $\quad |T_{s_1} \cap T_{s_2}| < \omega$

$s_1 \neq s_2$ $\quad$ in particular $T_{s_1} \neq T_{s_2}$

${}^{\kappa}2 \to [\omega]^{\omega}$

$\Rightarrow 2^{\kappa} \leq 2^{\omega}$

Construct $(T_s \mid s \in {}^{\leq\kappa}2)$ by induction:

$T_{\emptyset} := \omega$

If T_s defined set $T_{s^\frown\langle 0\rangle}$ and $T_{s^\frown\langle 1\rangle}$ as arbitrary infinite subsets of T_s which are disjoint.

If length of s is a limit λ then $\lambda \leq \kappa < \underline{t}$ and set T_s as pseudo-intersection of $\{T_{s \restriction \alpha} \mid \alpha < \lambda\}$.

$\square$

Cor. $\underline{t} \leq \mathrm{cof}(2^\omega)$

Proof: $\kappa < \mathrm{cof}(2^\kappa) = \mathrm{cof}(2^\omega)$ for $\kappa < \underline{t}$. $\square$

Def. $f, g \in {}^\omega\omega$. Define $f \leq^* g \;:\Longleftrightarrow\; |\{n \mid g(n) < f(n)\}| < \omega$

- Let $\mathcal{B} \subseteq {}^\omega\omega$.

  $\mathcal{B}$ is unbounded $:\Longleftrightarrow$ for all $g \in {}^\omega\omega$ there is $f \in \mathcal{B}$ s.t. $f \not\leq^* g$.

- $\underline{b} := \min\{|\mathcal{B}| \mid \mathcal{B} \subseteq {}^\omega\omega$ is an unbounded family$\}$.

Fact $\underline{t} \leq \underline{b}$

Prop Suppose $\lambda < \mathfrak{t}$ and $(T_\alpha \mid \alpha < \lambda)$ a decreasing $\subseteq^*$-chain, $T_\alpha \subseteq \mathbb{Q}$ all dense.
Then there is $X \subseteq \mathbb{Q}$ s.t. $X \subseteq^* T_\alpha$ $\forall \alpha < \lambda$ and X is dense in $\mathbb{Q}$.

Proof: For every interval I with rational endpoints consider $(T_\alpha \cap I \mid \alpha < \lambda)$, $T_\alpha \cap I$ inf.
Since $\lambda < \mathfrak{t}$ take $Y_I \subseteq I$ s.t.

$$Y_I \subseteq^* T_\alpha \cap I \quad \forall \alpha < \lambda, \quad |Y_I| \geq \omega$$

Let $(y_{I,n} \mid n < \omega)$ be an enumeration of Y_I.
For $\alpha < \lambda$ let $f_\alpha(I) \in \omega$ s.t.

$$y_{I,n} \in T_\alpha \quad \forall n \geq f_\alpha(I)$$

$$f_\alpha : \{I \mid I \text{ rational Interval}\} \to \omega$$

$$\{f_\alpha \mid \alpha < \lambda\} \quad \lambda \leq \mathfrak{t} \leq \mathfrak{b}$$

$\Rightarrow$ there is $g : \{I \mid \text{rat.}\} \to \omega$
s.t. $f_\alpha \leq^* g$

$$X := \bigcup \{ y_{I,n} \mid n > g(I) \}$$

- X is dense
- $X \subseteq^* T_\alpha$.

If $x \in X \setminus T_\alpha$ then $x = y_{I,n}$ with $n < f_\alpha(I)$
$x \in X \Rightarrow n > g(I)$ $\quad f_\alpha \leq^* g$
can happen only finitely many times. $\square$

Fact $\underline{t} \leq \mathrm{add}(\mathcal{M}) = \min\{|I| \mid$ there are meager set M_i, $i \in I$ s.t. $\bigcup M_i$ is not meager$\}$

Fact $\underline{b}(\omega_1) = \underline{t}(\mathrm{Club}(\omega_1), \underline{c}^*)$.

Prop. $\mathfrak{t} \leq \mathrm{add}(\mathcal{M})$

Proof: Let $\kappa < \mathfrak{t}$, show $\kappa < \mathrm{add}(\mathcal{M})$.

Enough to show: If G_α, $\alpha < \kappa$ open dense the $\bigcap_{\alpha<\kappa} G_\alpha$ is comeager

$\ulcorner$ M_α, $\alpha < \kappa$ meager $\Rightarrow \mathbb{R} \setminus M_\alpha$ contains G_δ-set G_α' which is dense.

$$G_\alpha' = \bigcap_{\gamma<\omega} G_{\alpha\gamma} \begin{array}{l} \text{open} \\ \text{dense} \end{array}$$

$$M_\alpha \subseteq \mathbb{R} \setminus G_\alpha'$$

$$\bigcup M_\alpha \subseteq \bigcup \mathbb{R} \setminus G_\alpha' = \mathbb{R} \setminus \bigcap G_\alpha' = \mathbb{R} \setminus \bigcap_{\alpha,\gamma} G_{\alpha\gamma}$$

$\lrcorner$

G_α, $\alpha<\kappa$ open dense

Define $(T_\alpha \mid \alpha \leq \kappa)$ dense subsets of $\mathbb{Q}$

$T_0 := \mathbb{Q}$

If $(T_\alpha \mid \alpha < \lambda)$ λ limit let $T_\lambda \subseteq \mathbb{Q}$ with $T_\lambda \subseteq^* T_\alpha$ $\alpha<\lambda\leq\kappa$ and T_λ dense in $\mathbb{Q}$.

(Prop from before)

If $(T_\alpha \mid \alpha \leq \beta)$ set. $T_{\beta+1} := T_\beta \cap G_\beta$.

$T_{\beta+1}$ is dense

For $\alpha < \kappa$ let

$$f_\alpha : T_\kappa \to \omega, \quad f_\alpha(t) := \begin{cases} \text{some } n \text{ with } (t-\frac{1}{n}, t+\frac{1}{n}) \subseteq G_\alpha \text{ if there is one} \\ 0 \quad \text{otherwise} \end{cases}$$

$$T_\kappa \subseteq^* T_{\alpha+1} = T_\alpha \cap G_\alpha \subseteq G_\alpha$$

Take $g : T_\kappa \to \omega$ with $f_\alpha \leq^* g$ $\forall \alpha<\kappa$

since $\kappa < \underline{t} \leq \underline{b}$

Define $F \subseteq T_\kappa$ finite

$$U_F := \bigcup_{t \in T_\kappa \setminus F} \left(t - \frac{1}{g(t)} , t + \frac{1}{g(t)} \right)$$

U_F open

and dense since $T_\kappa \subseteq^* U_F$

$$\bigcap_{F \subseteq T_\kappa \text{ fin.}} U_F \quad \text{comeager}$$

Notice: $$\bigcap_{F \subseteq T_\kappa \text{ fin.}} U_F \subseteq \bigcap_{\alpha < \kappa} C_{\delta\alpha}.$$

For every $\alpha < \kappa$ there are only fin. many $t \in T_\kappa \setminus C_{\delta\alpha}$ und $t \in T_\kappa$ with $g(t) < f_\alpha(t)$.

$\square$.

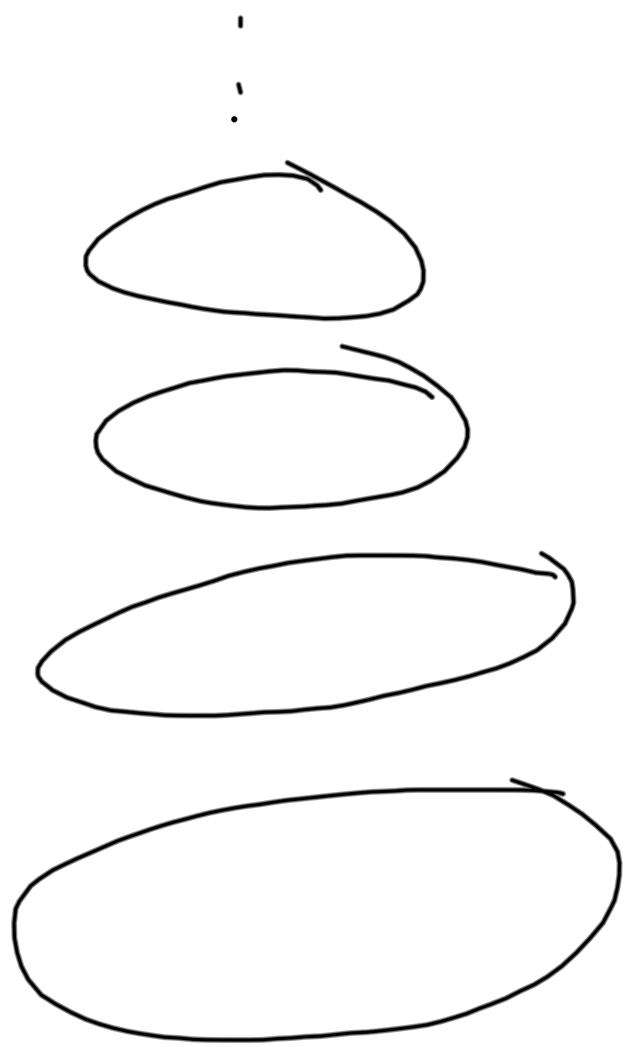
NEW FOUNDATIONS

math.boisestate.edu/~holmes/holmes/nfdoc.pdf

Extensionality

Comprehension

$\{z \mid z \notin z\}$



$x_7 \in y_8 \wedge y_8 = z_8$