Def Assume \( A, B \) are sets. Define
\[
\begin{align*}
A & \models B & \iff & A \setminus B \text{ and } B \setminus A \text{ are finite} \\
A & \preceq B & \iff & A \setminus B \text{ is finite}
\end{align*}
\]

Remark \((\mathcal{P}(w), \preceq^*)\) is a preorder
\((\mathcal{P}(w)/\sim^*, \preceq^*)\) is a partial order.
Def: Let $F$ be a family of sets. A set $A$ is a **pseudo-intersection** of $F$ iff $A =^* F$ for all $F \in F$, and $|A| \geq \omega$.

**Def:** $(T_\alpha \mid \alpha < \lambda)$ is called a tower if

1. $T_\alpha \in [\omega]^{\omega}$ for all $\alpha < \lambda$,
2. $T_\beta \subseteq^* T_\alpha$ for all $\alpha < \beta < \lambda$,
3. $\{T_\alpha \mid \alpha < \lambda\}$ does not have any pseudo-intersection.

Set $t := \min \{ \lambda \in \text{Ord} \mid \text{there is a tower } (T_\alpha \mid \alpha < t) \text{ of length } \lambda \}$

and call it the **tower number**.
Prop \( \hat{\tau} \) is a regular uncountable cardinal.

Proof: \( \hat{\tau} \) is not singular, since cofinal subsequence of a tower is a tower as well.

\[ \hat{\tau} = c \cdot (\hat{\tau}) \]

Let \( (T_n \mid n < \omega) \) be a sequence fulfilling (1) and (2). Show that it fails on (3).

Define \( x_0 \in T_0 \).

If \( x_m \) for \( m < n \) is defined, then take \( x_n \in \bigcap_{m<n} T_m \) (is inf.) \( x_n \neq x_m \) for \( m < n \).

\[ \{ x_n \mid n < \omega \} \preceq T_n \] for all \( n < \omega \). □
Fact: $\text{MA}(k) \Rightarrow 2^k = 2^\omega$.

Thm: If $w \leq k < \omega$ then $2^k = 2^\omega$.

Proof: $2^\omega \leq 2^k$ is clear.

Show $2^k \leq 2^\omega$.

We construct $(T_s \mid s \in 2^k)$ along the full binary tree $2^k$ of height $k+1$ with the properties:

1. $T_s \in [\omega]^w$ for all $s \in 2^k$,

2. If $s_1$ initial segment of $s_2$, then $T_{s_1} \preceq^* T_{s_2}$,

3. If neither $s_1$ extends $s_2$, nor $s_2$ extends $s_1$, then $T_{s_1} \cap T_{s_2}$ is finite.
Given a sequence with the above properties then
\[ s_1, s_2 \in \mathbb{X} \]
\[ |T_{s_1} \cap T_{s_2}| < \omega \]
In particular, \( T_{s_1} \neq T_{s_2} \)
\[ \mathbb{X} \rightarrow \omega \mathbb{N} \]
\[ \Rightarrow \mathbb{X} \leq \mathbb{2}^\mathbb{N} \]

Construct \( (T_s | s \in \mathbb{X}) \) by induction:
\[ T_\emptyset := \omega \]
If \( T_s \) defined set \( T_{s^{<\lambda}} \) and \( T_{s^{\lambda}} \)
\[ \text{as arbitrary infinite subsets of } T_s \]
which are disjoint.
If length of \( s \) is a limit \( \lambda \)
then \( \lambda < \mathfrak{t} \) and set \( T_s \)
\[ \text{as pseudo-intersection of } \bigcap_{\alpha < \lambda} T_{s_{\alpha}} \]
Cor: \( t \leq \text{cof}(2^\omega) \)

**Proof:** \( k < \text{cof}(2^\omega) = \text{cof}(2^\omega) \) for \( k \leq t \). \( \square \)

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**Def.** \( f, g \in \omega^\omega \). Define \( f \leq^* g : \iff |\{ n \mid g(n) < f(n)\}| < \omega \)

- Let \( B \subseteq \omega^\omega \).
- \( B \) is unbounded \( : \iff \) for all \( g \in \omega^\omega \) there is \( f \in B \) s.t. \( f \neq g \).
- \( b := \min \{ |B| \mid B \subseteq \omega^\omega \text{ is an unbounded family} \} \)

**Fact** \( t \leq b \)
Prop
Suppose \( \lambda < t \) and \((T_\lambda I | x < \lambda)\) a decreasing \( \leq^* \)-chain, \( T_\lambda \subseteq Q \) all dense.

Then there is \( X \subseteq Q \) s.t. \( X \preceq T_\lambda \forall x \lambda \) and \( X \) is dense in \( Q \).

Proof: For every interval \( I \) with rational endpoints consider \((T_\lambda \cap I | x < \lambda)\), \( T_\lambda \cap I \) inf. Since \( \lambda < t \) take \( y_I \subseteq I \) s.t.

\[ y_I \subseteq T_\lambda \cap I \forall x < \lambda, \ |y_I| \geq \omega \]

Let \( (y_{x_I} \cap I | x < \omega) \) be an enumeration of \( y_I \).

For \( x < \lambda \) let \( f_x(\cdot) \subseteq I \omega \) s.t.

\[ y_{x_I} \subseteq T_\lambda \quad \forall \ n \geq f_x(\cdot) \]

\( \forall \ I \) rational Interval \( ? \rightarrow \omega \)

\[ \{ f_x(\cdot) \ \mid \ x < \lambda \} \quad t \leq t \leq b \]

\[ \Rightarrow \ there \ is \ g: \ |I| \rightarrow \omega \]

s.t. \( f_x \leq g \)

\[ X := \{ y_{x_I, n} \mid n \geq g(\lambda) \} \]

\( X \) is dense

\[ X \preceq T_\lambda \]

If \( x \in X \cup T_\lambda \) then \( x = y_{x_n} \) with \( n < f_x(\lambda) \)

\[ x \in X \Rightarrow n \geq g(\lambda) \]

Can happen only finitely many times. \( \square \)
Fact \[ \exists \delta \leq \text{add}( \mathcal{M}) = \min \{ |\mathcal{I}| \mid \text{there are meager sets } M_i, i \in \mathcal{I} \text{ s.t. } \bigcup M_i \text{ is not meager} \} \]

Fact \[ d(\omega_n) = \pm(\text{Club}(\omega), c^*) \]
Prop. \( t \leq \text{add}(M) \)

Proof: Let \( k < t \), show \( k \leq \text{add}(M) \).

Enough to show: If \( \bigcap C_{\alpha} \) is open dense, then \( M_{\alpha} \) is meager.

\[
C_{\alpha} = \bigcap_{\delta < \alpha} C_{\delta} \text{ is dense.}
\]

\[
M_{\alpha} \subset R \setminus C_{\alpha}^1
\]

\[
UM_{\alpha} \subset U(R \setminus C_{\alpha}) = R \setminus \bigcap C_{\alpha} = R \setminus \bigcap C_{\alpha}^1
\]

\[\blacksquare\]
Consider open dense

Define \((T_\alpha \mid \alpha \leq k)\) dense subset of \(\mathbb{R}\)

\[ T_0 := \mathbb{R} \]

If \((T_\alpha \mid \alpha < \lambda)\) a limit set \(\\lambda < \lambda \leq k\) and \(T_\alpha\) dense in \(\mathbb{R}\).

If \((T_\alpha \mid \alpha \leq \beta)\) set \(T_{\beta+1} := T_\beta \cap C_\beta\).

\(T_{\beta+1}\) is dense.

For \(\alpha < k\) let

\[ f_\alpha : T_\alpha \to \mathbb{R}, \quad f_\alpha (t) := \left\{ \begin{array}{ll} \text{some } \eta \text{ with } (t - \frac{1}{n}, t + \frac{1}{n}) \subseteq C_{\eta} & \text{if there is one} \\ 0 & \text{otherwise} \end{array} \right. \]

\(T_\alpha \subseteq T_{\alpha+1} = T_\alpha \cap C_\alpha \subseteq C_\alpha\)

Take \(g : T_k \to \mathbb{R}\) with \(f_\alpha \leq g\) \(\forall \alpha < k\)

since \(k < t \leq b\)
Define $F \in T_k$ finite

$$U_F := \bigcup_{t \in T_k \setminus F} \left( + \frac{1}{y(a)} , t + \frac{1}{y(a)} \right)$$

$U_F$ open

and dense since $T_k \subseteq U_F$

$$\bigcap_{F \subseteq T_k} U_F$$

comeager

Notice:

$$\bigcap_{F \subseteq T_k} U_F \subseteq \bigcap_{a \in \mathbb{N}} C_{g,a}$$

For every $a \in \mathbb{N}$ there are only finitely many $t \in T_k \setminus C_{g,a}$ and $t \in T_k$ with $y(a) < \phi_x(t)$.
NEW FOUNDATIONS
Extensionality
Comprehension

\{ z \mid z \in x \}

\forall x, y, z (x \in y \land y \in z \Rightarrow x \in z)

x \in y \land y \in z \Rightarrow x \in z