

Arithmetical closures of a hierarchy of prewellorderings under AD

In ZFC: There are far many "nice properties" of subsets of the reals we can find counterexamples.

Example:

- Lebesgue measurability
- perfect set property
- Baire property (A has Baire property iff $\exists U \subseteq \mathbb{R}$ open s.t. $A \Delta U$ is meager)

Question: Can we avoid these counterexamples by restricting "complexity" of sets considered.

Pass from \mathbb{R} to ω^ω (Base space).

$$\omega^\omega \cong \mathbb{R} \setminus \mathbb{Q}.$$

Continuous fctns: $\omega^\omega \rightarrow \omega^\omega$ are just those functions f s.t. there is a fctn. $\sigma_f: \omega^{<\omega} \rightarrow \omega^{<\omega}$ s.t.

$$(1) \forall t, s \in \omega^{<\omega}, t \subseteq s \Rightarrow \sigma_f(t) \subseteq \sigma_f(s).$$

$$(2) \forall x \in \omega^\omega: \lim_{n \rightarrow \infty} L_n(\sigma(x \upharpoonright_n)) = \infty.$$

$$(3) \forall x \in \omega^\omega: f(x) = \bigcup_{n \in \omega} \sigma(x \upharpoonright_n).$$

If we replace (2) by (2'):

$$(2') \forall \eta \forall x \in \omega^\omega: L_\eta(\sigma(x \upharpoonright_\eta)) = \eta$$

then we get the Lipschitz fctns.

What can we say about the structure of the
Wadge hierarchy? I, ZFC: Not much useful.

Games and Determinacy

Let $A \subseteq \omega^\omega$. Then $G(A)$ is the following game
of two players playing natural numbers at each turn:

I $x_0 \quad x_2 \quad x_4 \quad \dots \quad \langle x_i \mid i \in \omega \rangle \in \omega^\omega$
 II $x_1 \quad x_3$

I wins $\Leftrightarrow \langle x_i \mid i \in \omega \rangle \in A$

II wins otherwise.

Def. A v.s. σ for I in $G(A)$ is a map
 $\sigma: \omega < \omega \longrightarrow \omega$ s.t. I always wins $G(A)$
 when reacting to a partial play of player II
 with playing $\sigma(s)$ throughout the match.

$$\left[\begin{array}{cccc} \text{I} & \sigma(\emptyset) & \sigma(\langle x_0 \rangle) & \sigma(\langle x_0, x_1 \rangle) & \dots \\ \text{II} & x_0 & & x_1 & \end{array} \right]$$

A set A is determined iff either I or II has
 a v.s. in $G(A)$.

BD expresses that all all Borel sets are determined

PD express that all projective sets are determined

AD expresses that all sets are determined.

$G_L(A, B)$ for $A \in \mathcal{U}^\omega$ is the following game:

I $x_0 \quad x_1 \quad x_2 \quad \dots \rightarrow x \in \mathcal{U}^\omega$

II $y_0 \quad y_1 \quad y_2 \quad \dots \rightarrow y \in \mathcal{U}^\omega$

Player II wins if $x \in A \Leftrightarrow y \in B$.

Lemma $A \leq_L B \iff \text{II has a w.s. in } G_L(A, B).$

$G_U(A, B)$ is the same game as $G_L(A, B)$ only that Player II can pass, but loses if she doesn't play natural number infinitely often.

Lemma $A \leq_U B \iff \text{II has a u.s. in } G_U(A, B).$

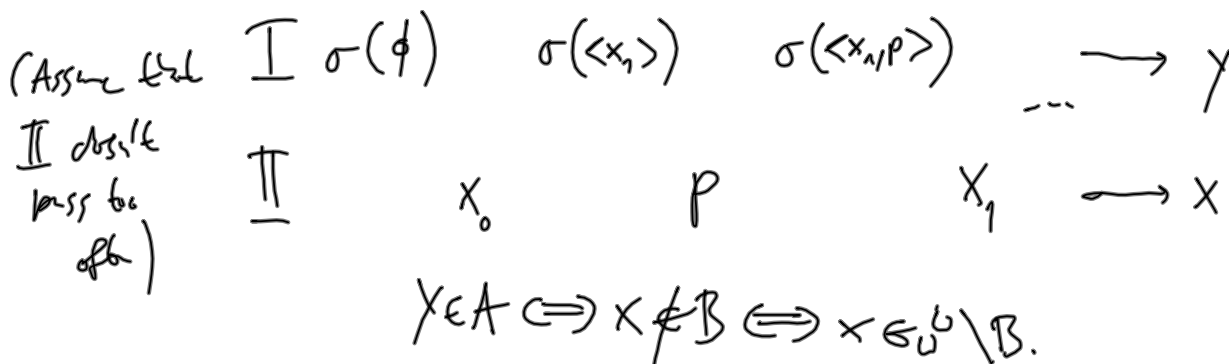
Thm (Vadze's Lemma).

Π is a boldface partition.
 If every $A \in \Pi$ is determined,
 then we have for any $A, B \in \Pi$:

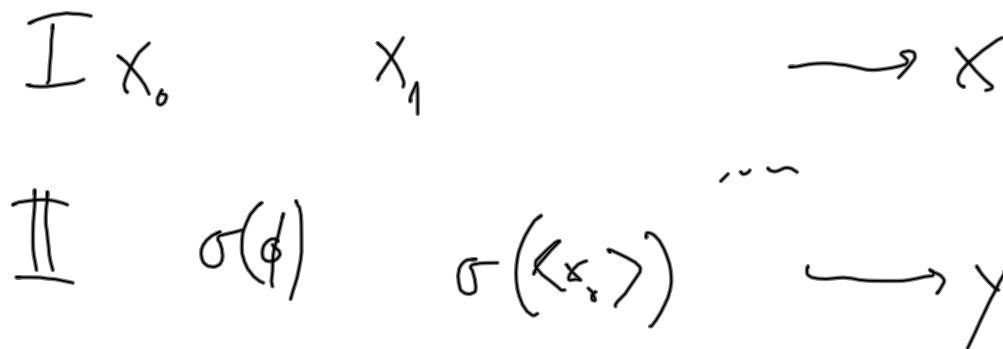
$$A \leq_w B \vee \omega \setminus B \leq_L A.$$

$\Pi \subseteq \mathcal{P}(U)$ is called a boldface partition iff it closed under \leq_w .

Proof: Assume Π doesn't sat $G_w(A, B)$. So I has a u.s. σ :



So in $G_L(\omega \setminus B, A)$ let Π play as follows:



$\gamma \in A \Leftrightarrow x \in \omega \setminus B.$



Thm (Mantel-Morik).

Assume AD + a bit of choice [DCCR].

$(\mathcal{P}(\omega) / \equiv_{\omega}, \leq_{\omega})$ and $(\mathcal{P}(\omega) / \equiv_{\omega}, \leq_{\omega})$
are well-founded.

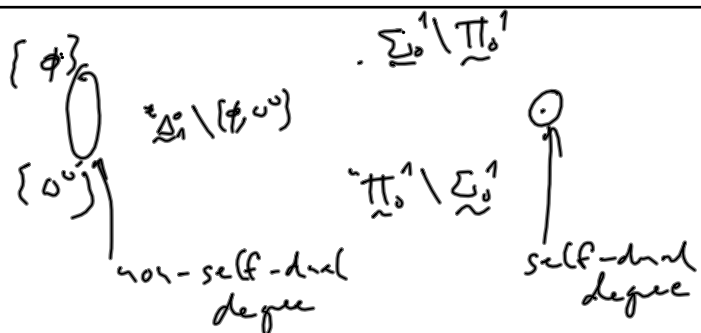
AD and AC are incompatible.

$ZF + AD \vdash$ all $A \subseteq \mathbb{R}$ have the Baire property

· \rightarrow p.s.p.

· are Lebesgue measurable.

$ZF + AC \vdash BD$, $ZF + AD \vdash AC_{\omega}(\mathbb{R})$



$$\textcircled{H} = \sup \{ \alpha \mid \exists f: 0^u \rightarrow \alpha \}$$

$$\text{ZFC} \vdash \textcircled{H} = (2^{\aleph_0})^+$$