# Arithmetical closures of a hierarchy of prewellorderings under AD

In ZFC: There are for many "nice properties" of subsets of the reals we can find counterexamples.

Example:
- Lebesgue measurability
- perfect set property
- Baire property (A has Baire property iff $\exists U \subseteq \mathbb{R}$ open s.t. $A \triangle U$ is meager)

Question: Can we avoid these counterexamples by restricting "complexity" of sets considered.

Pass from $\mathbb{R}$ to $\omega^\omega$ (Baire space).

$$\omega^\omega \cong \mathbb{R} \setminus \mathbb{Q}.$$

Continuous fctions $: \omega^\omega \longrightarrow \omega^\omega$ are just those functions $f$ s.t. there is a fct. $\sigma_f : \omega^{<\omega} \longrightarrow \omega^{<\omega}$ s.t.

(1) $\forall t, s \in \omega^{<\omega},\ t \subseteq s \Longrightarrow \sigma_f(t) \subseteq \sigma_f(s)$.

(2) $\forall x \in \omega^\omega : \lim_{n \to \infty} \mathrm{lh}(\sigma(x \restriction n)) = \infty$.

(3) $\forall x \in \omega^\omega : f(x) = \bigcup_{n \in \omega} \sigma(x \restriction n)$.

If we replace (2) by (2'):

(2') $\forall n \forall x \in \omega^\omega : \mathrm{lh}(\sigma(x \restriction n)) = n$

then we get the <u>Lipschitz fcts.</u>

<u>Def:</u> $A, B \subseteq \omega^\omega$; then

$A \leq_W B \;:\Longleftrightarrow\; \exists f: \omega^\omega \to \omega^\omega$ continuous s.t.

$\forall x \in \omega^\omega \quad x \in A \Longleftrightarrow f(x) \in B.$

$A \leq_L B \;:\Longleftrightarrow\; \exists g:$ — " — Lipschitz s.t.

— " — $\quad x \in A \Longleftrightarrow g(x) \in B.$

Consider partial orderings

$$\left(\mathcal{P}(\omega^\omega)/\equiv_W, \leq_W\right), \text{ Wadge hierarchy}$$

$$\left(\mathcal{P}(\omega^\omega)/\equiv_L, \leq_L\right), \text{ Lipschitz hierarchy}$$

What can we say about the structure of the Wadge hierarchy? In ZFC: Not much useful.

## Games and Determinacy

Let $A \subseteq \omega^\omega$. Then $G(A)$ is the following game of two players playing natural numbers at each turn:

$$\begin{array}{lccccc} \mathrm{I} & x_0 & & x_2 & & x_4 \\ & & & & & & \dots \quad \langle x_i \mid i \in \omega \rangle \in \omega^\omega \\ \mathrm{II} & & x_1 & & x_3 & \end{array}$$

$\mathrm{I}$ wins $\iff \langle x_i \mid i \in \omega \rangle \in A$

$\mathrm{II}$ wins otherwise.

<u>Def.</u> A w.s. $\sigma$ for I in $G(A)$ is a map $\sigma : \omega^{<\omega} \longrightarrow \omega$ s.t. I always wins $G(A)$ when reacting to a partial play of player II with playing $\sigma(s)$ throughout the match.

$$\left[\begin{array}{lllll} \mathrm{I} & \sigma(\emptyset) & \sigma(\langle x_0 \rangle) & \sigma(\langle x_0, x_1 \rangle) & \dots \\ \mathrm{II} & x_0 & x_1 & & \end{array}\right]$$

A set $A$ is determined iff either I or II has a w.s. in $G(A)$.

BD expresses that all all Borel sets are determined

PD express that all projective sets are determined

AD expresses that <u>all</u> sets are determined.

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$G_L(A,B)$ for $A,B\subseteq\omega^\omega$ is the following game:

$$\begin{array}{llllll} \mathrm{I} & x_0 & & x_1 & & x_2 & \longrightarrow x\in\omega^\omega \\ & & & & \cdots & \\ \underline{\mathrm{II}} & & y_0 & & y_1 & & y_2 \longrightarrow y\in\omega^\omega \end{array}$$

Player II wins if $x\in A \Longleftrightarrow y\in B$.

<u>Lemma:</u> $A \leq_L B \iff$ II has a w.s. in $G_L(A,B)$.

$G_W(A,B)$ is the same game as $G_L(A,B)$ only that Player II can pass, but loses if she doesn't play natural numbers infinitely often.

<u>Lemma:</u> $A \leq_W B \iff$ II has a w.s. in $G_W(A,B)$.

$\underset{\sim}{\Gamma} \subseteq \mathcal{P}(\omega^\omega)$ is called a boldface pointclass iff it closed under $\le_W$.

<u>Thm</u> (Wadge's Lemma).

$\underset{\sim}{\Gamma}$ is a boldface pointclass.
If every $A \in \underset{\sim}{\Gamma}$ is determined,
then we have for any $A, B \in \underset{\sim}{\Gamma}$:

$$A \le_W B \vee \omega^\omega \setminus B \le_L A.$$

<u>Proof:</u> Assume II doesn't win $G_W(A,B)$. So I has a w.s. $\sigma$:

(Assume that II doesn't pass too often)

$$\begin{array}{lllllll} \mathrm{I} & \sigma(\emptyset) & \sigma(\langle x_1\rangle) & \sigma(\langle x_1, p\rangle) & \dots & \longrightarrow & y \\ \mathrm{II} & x_0 & p & x_1 & & \longrightarrow & x \end{array}$$

$$y \in A \iff x \notin B \iff x \in \omega^\omega \setminus B.$$

So in $G_L(\omega^\omega \setminus B, A)$ let II play as follows:

$$\begin{array}{lllll} \mathrm{I} & x_0 & x_1 & \dots & \longrightarrow x \\ \mathrm{II} & \sigma(\emptyset) & \sigma(\langle x_0\rangle) & & \longrightarrow y \end{array}$$

$$y \in A \iff x \in \omega^\omega \setminus B. \quad \boxtimes$$

Thm (Martin–Monk).
Assume AD + a bit of choice [$DC(\mathbb{R})$].

$$\left(\mathcal{P}(\omega^\omega)/\equiv_W, \leq_W\right) \text{ and } \left(\mathcal{P}(\omega^\omega)/\equiv_L, \leq_L\right)$$

are well-founded.

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AD and AC are incompatible.

ZF+AD $\vdash$ all $A \subseteq \mathbb{R}$ · have the Baire property
· — " — p.s.p.
· are Lebesgue measurable.

ZF+AC $\vdash$ BD, ZF+AD $\vdash$ $AC_\omega(\mathbb{R})$

$\{\emptyset\}$ $\quad$ $\{\omega^\omega\}$

"$\Delta^0_1 \setminus \{\emptyset, \omega^\omega\}$"

$\Sigma^1_0 \setminus \Pi^1_0$

"$\Pi^1_0 \setminus \Sigma^1_0$"

non-self-dual degree

self-dual degree

$$\Theta = \sup\{\alpha \mid \exists f : \omega^\omega \twoheadrightarrow \alpha\}$$

$$ZFC \vdash \Theta = (2^{\aleph_0})^+$$