DEF: $(\mathbb{P}, \leq)$ is a partial order. We say that G is a GENERIC FILTER for $\{D : D \in \mathcal{D}\}$ iff:

1. $\forall p, q \in G$, $\exists r \leq p, q$ s.t. $r \in G$
2. If $q \in G$ and $p \geq q$ then $p \in G$

} G is a FILTER

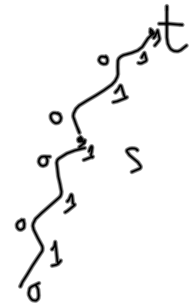
3. $\forall D \in \mathcal{D}$, $G \cap D \neq \emptyset$.

Who is $\{D : D \in \mathcal{D}\}$? It is a family of DENSE subsets of $\mathbb{P}$, ie,

$D \subseteq \mathbb{P}$ is <u>DENSE</u> iff $\forall p \in \mathbb{P}\ \exists q \in D$ s.t. $q \leq p$.

COHEN FORCING: $\mathbb{P}$ is simply the set of finite sequences of 0's and 1's.

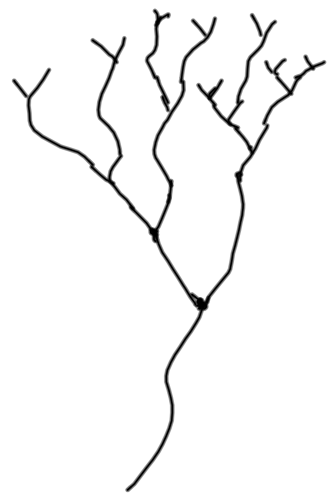
$t \leq s \iff t$ extends s



REMARK 1 $D_m := \{ t \in 2^{<\omega} : |t| \geq m \}$ is DENSE

DEF: Let $\mathbb{S}$ be the partial order consisting of perfect trees ordered by inclusion, i.e.,

$$T \leq S \iff T \subseteq S.$$



<u>LEMMA</u>: $\mathbb{S}$ preserves $\aleph_1$.

<u>PROOF</u>: <u>AIM</u>: Given $A \in M[G]$ countable seq. of ordinals $\exists B \in M$ countable seq. of ordinals s.t. $B \supseteq A$.

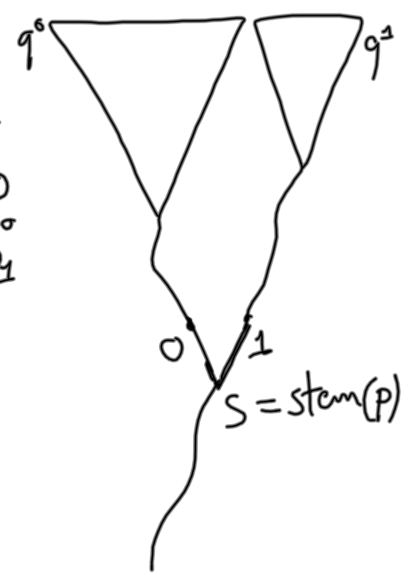
$F : \omega \to On$ and $\operatorname{ran}(F) = A$.

Fix $p \in \mathbb{S}$ $\xrightarrow{\text{AIM}}$ Build $q \in \mathbb{S}$, $q \le p$ and $\overset{\in M}{B}$ s.t. $q \Vdash \operatorname{ran}(F) \subseteq B$.

- pick $p_{s^\frown 0}, p_{s^\frown 1}$

and take $q^0 \le p_{s^\frown 0}, b_0^0 \in \mathrm{On}$ s.t. $q^0 \Vdash \dot{F}(0) = b_0^0$

$q^1 \le p_{s^\frown 1}, b_1^0 \in \mathrm{On}$ s.t. $q^1 \Vdash \dot{F}(0) = b_1^0$



$p^0 := q^0 \cup q^1 \in \mathbb{S}$ and $\mathrm{stem}(p_0) = s$

$B_0 := \{b_0, b_1\}$

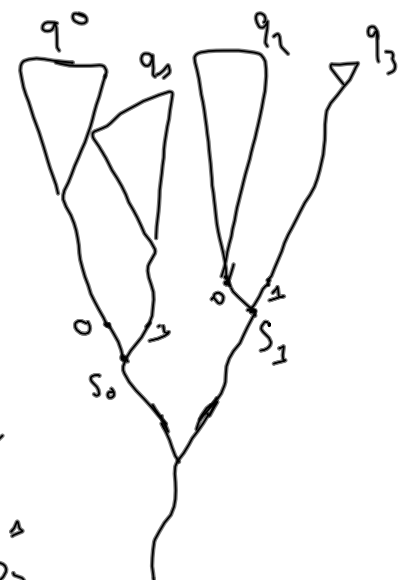
May 21, 2014

$p^0_{s_0^\frown 0} \geq q^0 , b^1_0 \qquad q^0 \Vdash F(1)=b^1_0$

$p^0_{s_0^\frown 1} \geq q^1 , b^1_1 \qquad q^1 \Vdash F(1)=b^1_1$

$p^0_{s_1^\frown 0} \geq q^2 , b^1_2 \qquad q^2 \Vdash F(1)=b^1_2$

$p^0_{s_1^\frown 1} \geq q^3 , b^1_3 \qquad q^3 \Vdash F(1)=b^1_3$



$p^1 := q^0 \cup q^1 \cup q^2 \cup q^3$ and $B^1 = \{ b^1_0, b^1_1, b^1_2, b^1_3 \}$

By induction, we proceed analogously, and we get

$\{p^n : n\in\omega\}$ of trees in $\mathbb{S}$ s.t.

- $p^{n+1} \leq p^n$
- p^{n+1} and p^n have the same Kth-splitnodes, for $K \leq n+1$

And so $\bigcap_{n\in\omega} p_n =: q \in \mathbb{S}$

Further let $B = \bigcup_{n\in\omega} B_n$

CLAIM:

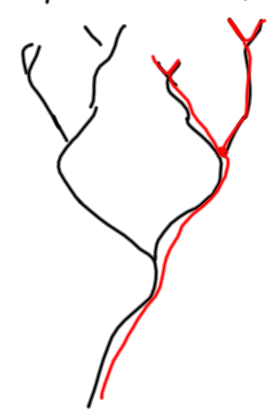
$$q \Vdash \operatorname{ran}(\dot{F}) \subseteq B$$

$$q \Vdash \dot{F}(0) = b_0^0 \vee \dot{F}(1) = b_1^0$$

$$q \Vdash \bigvee_{j\leq 3} \dot{F}(1) = b_j^1$$

$\vdots$

$\mathbb{S}$:= forcing consisting of perfect trees, ordered by $\subseteq$.



LAST TIME: We proved that

$\mathbb{S}$ preserves $\aleph_1$.

THM. $\mathbb{S}$ is ω^ω-bounding.

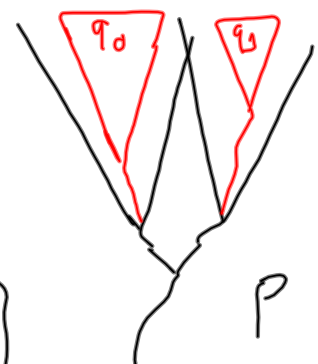
$\hookrightarrow$: i.e., $\forall x \in \omega^\omega \cap M[G]\ \exists y \in \omega^\omega \cap M$ s.t. $\forall^\infty_{n\in\omega}$ $(x(n) \le y(n))$.

PROOF: AIM: Start $p \in \mathbb{S}$, $\dot{x} \in \omega^\omega \cap M[G]$ $\longrightarrow$ we build $q \le p$, $q \in \mathbb{S}$, $y \in \omega^\omega \cap V$ s.t. $q \Vdash \forall^\infty_{m \in \omega} (\dot{x}(m) \le y(m))$.

STEP 0: let $t = \text{stem}(p)$. Pick

$q^0 \le p_{t^\frown 0}$ and $b^0 \in \omega$ s.t. $q^0 \Vdash \dot{x}(0) = b^0$

$q^1 \le p_{t^\frown 1}$ and $b^1 \in \omega$ s.t. $q^1 \Vdash \dot{x}(0) = b^1$



$p^0 := q^0 \cup q^1$. NB: $p^0 \Vdash \dot{x}(0) \in \{b^0, b^1\}$

$y(0) = \max\{b^0, b^1\} + 1$

We can analogously find $p^n \le_n p^{n-1}$ s.t. $p^n \Vdash \dot{x}(n) \in \{b^j : j < 2^{n+1}\}$

and $y(n) = \max\{b^j : j < 2^{n+1}\} + 1$

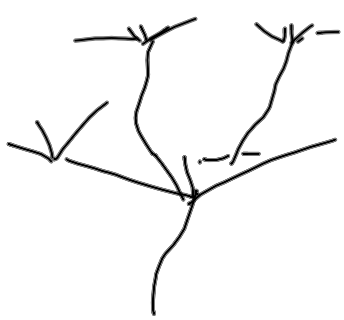
As last time, put $p := \bigcap_{n \in \omega} p_n \in \mathbb{S}$

and $p \Vdash \forall n \in \omega \, (\dot{x}(n) < y(n))$.

$\blacksquare$

OTHER EXAMPLE:

MILLER FORCING $\mathbb{M}$:= poset consisting of perfect trees $p \subseteq \omega^{<\omega}$ s.t. $\forall t \in p$, $t \in$ splitting $\Rightarrow \exists$ infinitely many $n \in \omega$ s.t. $t^\frown n \in p$



- $\mathbb{M}$ preserves $\aleph_1$. (by the "same" proof as for $\mathbb{S}$).

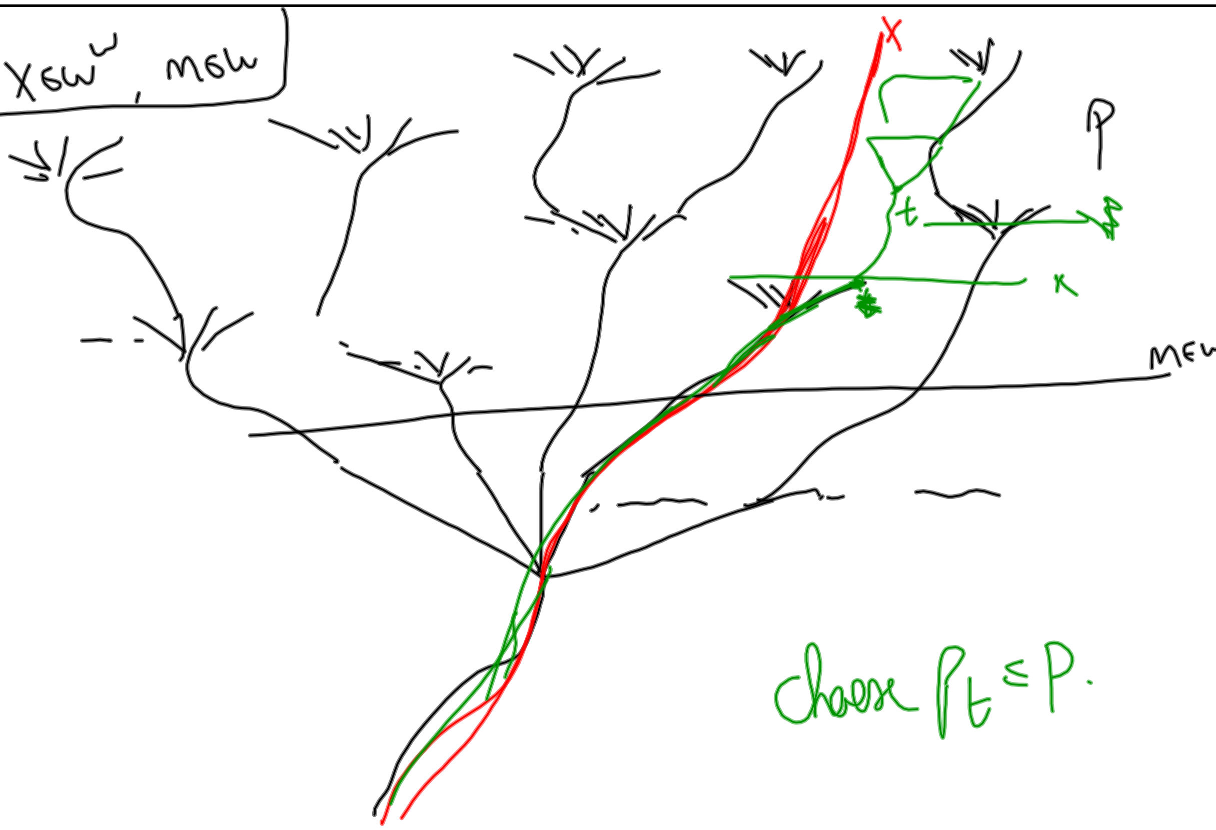
- <u>But</u> : $\underbrace{\mathbb{M} \text{ is } \underline{\text{not}}\ \omega^\omega\text{-bounding}}$.

↳ <u>PROOF</u>: It is enough to show that, for every $x \in \omega^\omega \cap M$ and $n \in \omega$,

$$D_{x,n} := \{ p \in \mathbb{M} : \exists k > n \text{ s.t. } |\text{stem}(p)| > k \wedge \text{stem}(p)(k) > x(k) \}.$$

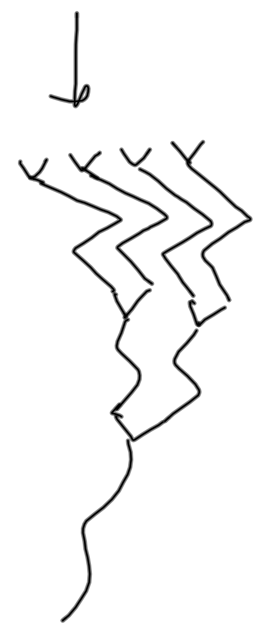
is dense.

$\boxed{\text{Fix } x \in \omega^\omega,\ m \in \omega}$



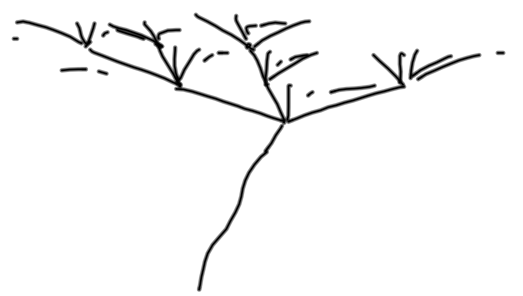
Choose $p_t \leq p$.

SILVER FORCING $\mathbb{V}$:= poset consisting of uniform perfect trees in $2^{<\omega}$



$\mathbb{V}$ preserves $\aleph_1$ and is ω^ω-bounding.

LAVER FORCING $\mathbb{L}$:



- $\mathbb{L}$ preserves $\aleph_1$

- $\mathbb{L}$ adds a dominating real.

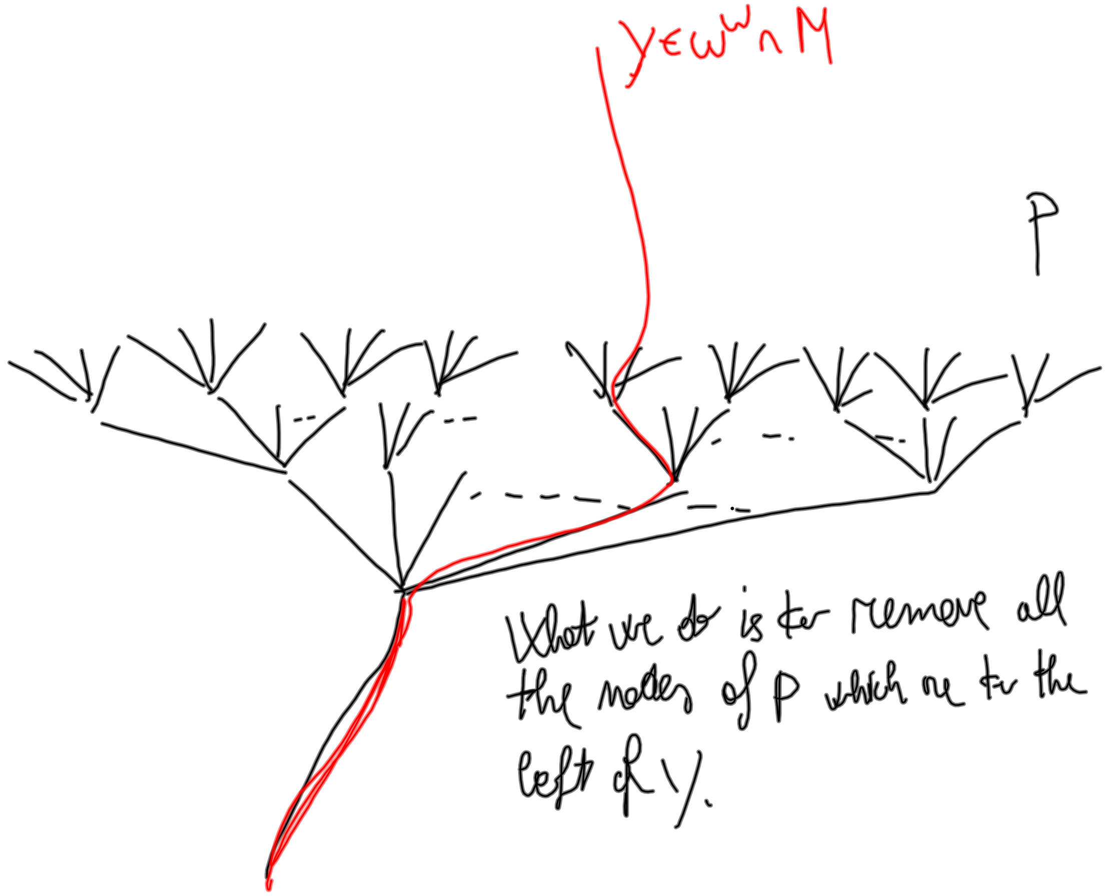
i.e., $\exists x \in \omega^\omega \cap M[G]$ s.t. $\forall y \in \omega^\omega \cap M$ $\forall^\infty n \, (y(n) \leq x(n))$.

May 21, 2014

PROOF: We have to check that, for every $y \in \omega^\omega \cap M$,

$$D_y := \{ p \in \mathbb{L} : \forall m > |\mathrm{stem}(p)| \; \forall t \in p \; (t(m) > y(m)). \}$$

is dense.

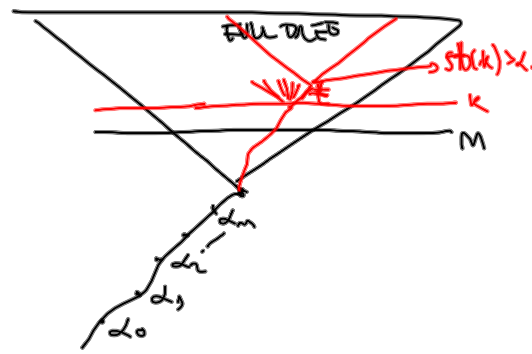


What we do is to remove all the nodes of P which are to the left of y.

<u>REMARK</u>: All tree-frames which we have seen are Π_1'-preserving.

- We want to see an example of a tree-frame which is not Π_1'-preserving.

Take $\mathbb{P}^{\cong \omega_1^{<\omega}}$ $\longrightarrow$



$(\omega_1 = \aleph_1)$

NOTE THAT:

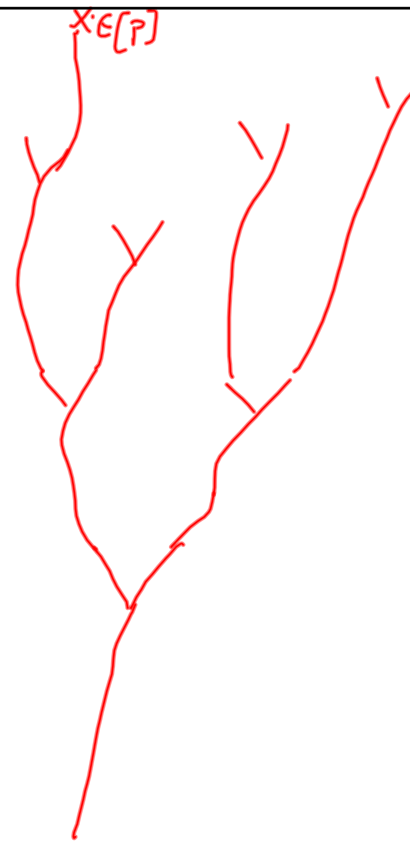
for every $\alpha<\omega_1$, for every $m\in\omega$,

$$D_{\alpha,m} := \left\{ p\in\mathbb{P} : \exists k>m \text{ st. } |\text{stem}(p)|>k \wedge \text{stem}(p)(k)>\alpha \right\}.$$

is dense.

GENERALISED SACKS FORCING

(perfect tree in $2^{<\kappa}$, with κ uncountable).



<u>REQUIREMENT</u>:

$\forall x \in [p]$, $\{\alpha < \kappa : x \upharpoonright \alpha \in \text{splitting}\}$ is a club.