DEF: \((P, \leq)\) is a partial order. We say that \(G\) is a **generic filter** if 
\[\forall D : D \in \mathcal{D}\]
\[\text{iff } \forall p \in G, \exists q \in p \text{ s.t. } q \leq p \]
\[\text{iff } q \in G \text{ and } p \trianglerighteq q \text{ then } p \in G \]
\[\forall D \in \mathcal{D}, \ G \cap D \neq \emptyset.\]

What is \(\{D : D \in \mathcal{D}\}\)? It is a family of **dense subsets** of \(P\), i.e., 
\(D \subseteq P\) is **dense** iff \(\forall p \in P \exists q \in D\) s.t. \(q \leq p\).
COWEN FORCING: \( P \) is simply the set of finite sequences of 0’s and 1’s.

\[ t \leq s \iff t \text{ extends } s \]

REMARK: \( D_m := \{ t \in 2^\omega : |t| \geq m \} \) is DENSE
DEF: let $S$ be the partial order consisting of perfect trees ordered by inclusion, i.e., $T \leq S \iff T \subseteq S$. 
**Lemma:** \( S \) preserves \( \mathfrak{M}_1 \).

**Proof:** aim: Given \( A \in \mathcal{M}(G) \) countable seq. of ordinals \( \exists B \in \mathcal{M} \) countable seq. of ordinals s.t. \( B \geq A \).

\( F : \omega \to \text{On} \) and \( \text{ran}(F) = A \).

Fix \( p \in S \). aim: Build \( q \in S, q \leq p \) and \( B \in \mathcal{M} \) s.t. \( q \upharpoonright \text{ran}(F) \leq B \).
pick $P^{-0}, P^{-1}$

and take $q^0 = P^{-0}, b_0 \in A, q^1 \downarrow F(0) = b_0^0$

$q^2 = P^{-1}, b_1 \in A, q^2 \downarrow F(0) = b_2^0$

$p^0 = q^0 \cup q^1 \in \mathcal{S}$ and $\text{stem}(p^0) = s$

$B_0 = \{b_0, b_1\}$
\[ P_{s_0}^0 = q^0, b_0 \quad q^0 \rightarrow F(b_0) = b_2 \]
\[ P_{s_1}^0 = q^3, b_1 \quad q^3 \rightarrow F(b_1) = b_3 \]
\[ P_{s_0}^1 = q^2, b_1 \quad q^2 \rightarrow F(b_1) = b_2 \]
\[ P_{s_1}^1 = q^3, b_3 \quad q^3 \rightarrow F(b_3) = b_3 \]

\[ P_1 = q^0, q^3, q^2, q^3 \quad \text{and} \quad B_1 = \{ b_0, b_1, b_3 \} \]
By induction, we proceed analogously, and we get

\{ P^m : m \in \mathbb{N} \} of trees in $\mathcal{S}$ s.t.

1. $P^{m+1} \leq P^m$
2. $P^{m+1}$ and $P^m$ have the same $K$th-splitmaker, for $K \leq m+1$

And so $\bigcap_{m=1}^\infty P_m = q \in \mathcal{S}$

Further let $B = \bigcup_{m=1}^\infty B_m$

**Claim:**

$q \vdash \text{Rom}(F) \leq B$

$q \vdash F(0) = b_0 \vee F(a) = b_j^0$

$q \vdash \forall j \leq 3, F(j) = b_j^3$

$\vdash \forall j \leq 3$
$S_i$: Family consisting of perfect trees, ordered by $\preceq$.

**Last time:** We proved that $S$ preserves $\mathbb{N}_1$.

**Today:** $S$ is $\omega$-bounding.

\[ \forall x \in \mathcal{W}^\omega \exists y \in \mathcal{W}^\omega \text{ s.t. } \forall m \in \mathbb{N} \quad (x(m) \equiv y(m)). \]
Proof: Let $s \leq \text{stem}(p)$. Pick

\[ q^0 \leq p^t_0 \quad \text{and} \quad b \in w \quad \text{st} \quad q^0 \mid b \cdot x(a) = b^0 \]
\[ q^1 \leq p^t_1 \quad \text{and} \quad b \in w \quad \text{st} \quad q^1 \mid b \cdot x(a) = b^1 \]

Let $p^0 = q^0 \cup q^1$. NB: $p^0 \mid x(a) \in \{b^0, b^1\}$

\[ Y(a) = \max\{b^0, b^1\} + 1 \]

We can analogously add $p^m \leq p^{m-1}$ st. $p^m \mid x(a) \in \{b^j : j < 2^m\}$

\[ Y(m) = \max\{b^j : j < 2^m\} + 1 \]
As last time, put \( p = \bigcap p_m \in S \) and
\[
P \Vdash \forall m \in \omega \, (x(m) < y(m)).
\]

**Other example:**

Miller Forcing \( \mathcal{M} := \) poset consisting of perfect trees \( p \in \mathcal{P}^\omega \)
s.t. \( A \) step, \( \mathcal{E} \) splitting \( \rightarrow \) infinitely
many new \( \sigma \).
M preserves $N_1$. (by the "seven" proof or by S).

But: $M$ is not $w$-bewildering.

Proof: It is enough to show that, for every $x \in w^* \cap M$

\[ D_{x,m} := \{ p \in M : \exists k > m \text{ s.t. } (\text{stem}(p)) > k \land \text{stem}(p)(k) > x(k) \} \]

is dense.
Fix $x_0^w, m_0^w$ 

Choose $p_t \in P$. 

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Silver forcing $V := \text{post counting of \text{uniform perfect trees in } \omega}$

$\mathbb{V}$ preserves $\text{PFA}_\omega$ and

is $\text{PFA}_\omega$-benign.
LAYER FORCING $L$:

- $L$ preserves $\mathcal{M}_2$,

- $L$ adds a dominating real.

\[ \text{Defining } \exists \omega, \exists x, \forall n < \omega, \forall y \in M \forall \omega (y(n) \leq x(n)). \]
Proof: We have to check that, for every $x \in x^u \cap M$,

$$D_y := \{ p < \Pi : \forall m > |\text{st}(p)| \forall t \in P \}
\begin{array}{l}
\quad (t(m) > y(m))
\end{array}$$

is dense.
What we do is to remove all the nodes of $P$ which are to the left of $Y$. 
Remark: All tree-brugs which we have seen are $\mathcal{N}_3$-preserving.

- We want to see an example of a tree-brug which is not $\mathcal{N}_3$-preserving.
Take $P = \omega_3^\omega$.

$(\omega_3 = \aleph_3)$

Note that for every $\alpha < \omega_3$, for any $m$, either $\delta_m : \{ \rho \in P : \exists k \geq m \text{ s.t. } \chi_k(\rho) \cap M \neq \emptyset \}$ is dense.
Generalized Stacks Forensics
(perfect tree in $2^k$, with $k$ countable).

Requirement:
$\forall x \in P$, $\{ \alpha < k : x \alpha \in \text{splitting} \}$ is a club.