XII Twelfth Lecture of The Forcing Method

9 July 2026

What happened so far...

We proved $\mathrm{Cons}(\mathrm{ZFC}) \Longrightarrow \mathrm{Cons}(\mathrm{ZFC} + \neg \mathrm{CH})$.

More importantly, we provide a construction method s.t.

if $M \models \mathrm{ZFC}$ is a ctm,
then there are $\mathbb{P}, G$ s.t.
$M[G] \models \mathrm{ZFC} + 2^{\aleph_0} \geq \aleph_2$ is a ctm.

Q. If $\aleph_1$ is not the size of the continuum, then what is it?

Lecture XI

Remark.
Determining $2^{\aleph_0}$ cannot be done without knowing something about M.

Further remark Note that replacing $Fn(\aleph_2^M \times \omega, 2)$ by $Fn(\aleph_n^M \times \omega, 2)$ gives an injection from $\aleph_n^M = \aleph_n^{M[G]}$ into $P(\omega)$ since the lemma about c.c.c. holds for $Fn(X,2)$ for arbitrary X.

The reason: This implies that we get models of

$$\boxed{2^{\aleph_0} \geq \aleph_n}$$

This implies that those are models of ZFC + $2^{\aleph_0} \geq \aleph_n$ and if M is such a model and $\mathbb{P} = Fn(\aleph_2^M \times \omega, 2)$, then in $M[G]$, $2^{\aleph_0}$ is still at least $\aleph_n$.

This means: Any answer to the question "What is $2^{\aleph_0}$ in $M[G]$?" must depend on M.

§23 Nice Names

Let $\mathbb{P}$ be any forcing.

Def. A name τ for a subset of ω is called a *nice name* if there is an $\mathcal{A} \in M$ s.t.

$$\mathcal{A} = \{A_n ; n \in \mathbb{N}\}$$

where A_n is an antichain in $\mathbb{P}$ and

$$\tau = \tau_{\mathcal{A}} = \{(\check{n}, p) ; p \in A_n\}.$$

NICE NAME THEOREM

Every subset of $\mathbb{N}$ has a nice name.

I.e., for all $A \subseteq \mathbb{N}$ s.t. $A \in M[G]$, there is $\mathcal{A} \in M$ s.t. $\tau_{\mathcal{A}}$ is a nice name and $A = \mathrm{val}(\tau_{\mathcal{A}}, G)$.

Proof. Let μ be a name for A (possibly not nice), i.e.,

$$A = \mathrm{val}(\mu, G).$$

Now, for each $n \in \mathbb{N}$ let

A_n be an antichain of elements p of $\mathbb{P}$ s.t.

$$p \Vdash \check{n} \in \mu$$

and A_n is maximal with these conditions.

[i.e., if $B \supsetneq A_n$ then either some $q \in B$ does not force $\check{n} \in \mu$ or B is not an antichain.]

The existence of these A_n uses AC/ZL.

Let $\mathcal{A} := \{A_n ; n \in \mathbb{N}\}$, so that $\tau_{\mathcal{A}}$ is a nice name.

CLAIM $\quad val(\tau_{\mathcal{A}}, G) = val(\mu, G) = A.$

"$\subseteq$": If $n \in val(\tau_{\mathcal{A}}, G)$, then there is $p \in G$ with $(\check{n}, p) \in \tau_{\mathcal{A}}$. Thus $p \in A_n$ (!), and therefore $p \Vdash \check{n} \in \mu$

So $n \in val(\mu, G) = A$.

"2". Let $n \in A = \mathrm{val}(\mu, G)$.

By the FT, get some $p \in G$ s.t.

$$p \Vdash \check{n} \in \mu.$$

Claim $A_n \cap G \neq \emptyset$.

[If $A_n \cap G = \emptyset$, then by "CTL (iii)", we find $q \in G$ s.t. $\forall r \in A_n \; q \perp r$.

Find $s \leq p, q$ s.t. $s \in G$ since G filter. Then $s \Vdash \check{n} \in \mu$ and s is still incompatible with all elts of A_n.

But then $A_n \cup \{s\}$ is an antichain strictly larger than A_n. Contradiction!]

If now $p \in A_n \cap G$, then $(n, p) \in \tau_{\alpha}$ and thus $n \in \mathrm{val}(\tau_{\alpha}, G)$.

q.e.d.

Lecture IX, p. 4

A crucial technical lemma

Def. $p \in \mathbb{P}$. A set $E \subseteq \mathbb{P}$ is called dense below p if $\forall q \leq p \; \exists r \leq q \; r \in E$.

Lemma Assume $G \subset \mathbb{P}$-gen./M, $E \subseteq \mathbb{P}$, $E \in M$. Then:

(i) If E is dense below p, $q \leq p$, then E is dense below q. [Trivial.]

(ii) If $\{r; E$ is dense below $r\}$ is dense below p, then E is dense below p. [Easy.]

(iii) Either $G \cap E \neq \emptyset$ or $\exists q \in G \; \forall r \in E \; r \perp q$. [Since $D = \{p; \exists r \in E \; p \leq r$ or $\forall r \in E \; p \perp r\}$.

I claim that D is dense. If $q \in D$, then either $\exists r \in E$ s.t. q and r are compatible. Find $q' \leq q, r$. Then $q' \in D$.

Therefore, there is $p \in G \cap D$. This witnesses the claim in (iii).]

(iv) If $p \in G$, E is dense below p, then $G \cap E \neq \emptyset$. [Suppose $G \cap E = \emptyset$. By (iii), find $q \in G$ s.t. $\forall r \in E \; r \perp q$. Find $q' \leq p, q$ s.t. $q' \in G$. Since E is dense below p and $q' \leq p$, find $r \in E$ s.t. $r \leq q' \leq q$. ↯] q.e.d.

Remark There is nothing special about $\mathbb{N}$: we can define nice names for any $X \in M$ and prove the corresponding nice name theorem.

IMPORTANCE OF THE NICE NAME THEOREM:

We can count nice names!

First, count the number of antichains. Say, $\mathbb{P}$ has the c.c.c. and

$$M \models |\mathbb{P}| = \mu$$

Then there are $(\mu^{\aleph_0})^M$ many antichains, and therefore

$$\left((\mu^{\aleph_0})^{\aleph_0}\right)^M$$ many nice names.

$$(\mu^{\aleph_0 \cdot \aleph_0})^M = (\mu^{\aleph_0})^M.$$

FACT If $M \models \mathbb{P}$ has c.cc. + $|\mathbb{P}| = \mu$, then $M \models$ there are $\mu^{\aleph_0}$ nice names

Corollary Under the assumptions of the Fact,

$$M[G] \models 2^{\aleph_0} \leq (\mu^{\aleph_0})^M.$$

Hausdorff Formula

A method to recursively calculate cardinal exponentiation on the basis of basic cases.

$$\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1}$$

Examples

$$\aleph_1^{\aleph_0} = \aleph_0^{\aleph_0} \cdot \aleph_1 = 2^{\aleph_0} \cdot \aleph_1 = 2^{\aleph_0}.$$

$$\aleph_2^{\aleph_0} = \aleph_1^{\aleph_0} \cdot \aleph_2 = 2^{\aleph_0} \cdot \aleph_2$$

$$\aleph_3^{\aleph_0} = \aleph_2^{\aleph_0} \cdot \aleph_3$$

$$= 2^{\aleph_0} \cdot \aleph_3$$

$$\aleph_\omega^{\aleph_0} = 2^{\aleph_0} \cdot \aleph_\omega$$

Apply this to $\mathbb{P} = Fn(\aleph_2^M \times \omega, 2)$.

We have $|\mathbb{P}| = \aleph_2$.

Thus, we get $M[G] \models$

$$2^{\aleph_0} \leq \aleph_2 \cdot (2^{\aleph_0})^M$$



Felix Hausdorff

Born	November 8, 1868 Breslau, Kingdom of Prussia (now Wrocław, Poland)
Died	January 26, 1942 (aged 73) Bonn, Germany

Corollary If $M \models 2^{\aleph_0} = \aleph_1$ or
$M \models 2^{\aleph_0} = \aleph_2$
and G is $Fn(\aleph_2^M \times \omega, 2)$-generic,
then $M[G] \models 2^{\aleph_0} = \aleph_2$.

Using the remark on p. 2:
$\mathbb{Q}_1 = Fn(\aleph_1^M \times \omega, 2)$ has c.c.c.
and size $\aleph_1^M$
Thus if $M \models CH$ and G is $\mathbb{Q}_1$-generic
then $M[G] \models 2^{\aleph_0} = \aleph_1$.

§ 24 The case of $\aleph_\omega$

Is it possible to get a model of $2^{\aleph_0} = \aleph_\omega$?

Answer: NO!

Theorem (König's Lemma.)

$$\aleph_\omega^{\aleph_0} > \aleph_\omega$$

Proof. We show this by "diagonalizing". Assume

$$F : \aleph_\omega \longrightarrow \aleph_\omega^{\aleph_0} = \{ f \mid f : \aleph_0 \longrightarrow \aleph_\omega \}$$

and prove that F is not a surjection.

This proves the claim.

Define $d : \aleph_0 \longrightarrow \aleph_\omega$

$$d(n) := \min\{ \beta \in \aleph_\omega \mid \forall \alpha < \aleph_n \ \beta \neq F(\alpha)(n) \}$$

I claim that $d \notin \mathrm{ran}(F)$. Suppose

$d = F(\delta)$ for some $\delta \in \aleph_\omega$.

Find n s.t. $\delta < \aleph_n$.

But then $d(n) \neq F(\delta)(n)$,

so $d \neq F(\delta)$.

q.e.d.

Gyula Kőnig



Born	16 December 1849 Győr, Kingdom of Hungary
Died	8 April 1913 (aged 63) Budapest, Austria-Hungary

Corollary $2^{\aleph_0} \neq \aleph_\omega$.

Proof. Assume it is. Then

$$2^{\aleph_0 \cdot \aleph_0} = (2^{\aleph_0})^{\aleph_0} = \aleph_\omega^{\aleph_0} > \aleph_\omega$$

$$\| \quad \text{KL}$$

$$2^{\aleph_0}$$

$$\|$$

$$\aleph_\omega$$

Contradiction

q.e.d.

By previous remarks, forcing with $\mathrm{Fn}(\aleph_\omega^M \times \omega, 2)$ gets

$$M[G] \models 2^{\aleph_0} \geq \aleph_\omega.$$

By König's Lemma, we know it can't be equal. So, forcing with $\mathrm{Fn}(\aleph_\omega^M \times \omega, 2)$ mysteriously adds more subsets than the natural proof gives.

Note that the upper bound given by the nice name theorem is

$$\aleph_\omega^{\aleph_0}$$

which by König is at least $\aleph_{\omega+1}$.

If you start with a model of GCH,
i.e., $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all α,
then one can calculate

$$\aleph_\omega^{\aleph_0} = \aleph_{\omega+1},$$

and so the nice names theorem gives
$\aleph_{\omega+1}$ as upper bound and hence

$$M[G] \models 2^{\aleph_0} = \aleph_{\omega+1}.$$

§25 Remarks on controlling $2^{\aleph_1}$.

The basic idea to add subsets of $\aleph_1$ would be

$$\mathbb{P} = Fn(k \times \aleph_1^M, 2).$$

In general $Fn(k \times \lambda, 2)$ will create by exactly the same arguments an injection from k into $\mathcal{P}(\lambda)$. Since all of these still have the ccc, we get

$$M[G] \models 2^\lambda \geq k \quad \text{in general.}$$

Of course, that can be obtained already by $Fn(k \times \omega, 2)$. So, the interesting question is:

will I get a model where

$2^{\aleph_0}$ is small, but 2^λ is large.

With the forcing $\mathbb{P}$ that is <u>not</u> the case: if G defines the generic function

$$f_G : k \times \aleph_1^M \longrightarrow 2,$$

then $f_G \restriction k \times \omega$ is a generic fn for $\mathbb{P}^* = Fn(k \times \omega, 2)$.

Therefore, if $M \models 2^{\aleph_0} = \aleph_1 + 2^{\aleph_1} = \aleph_2$

and $\mathbb{P} = Fn(\kappa \times \aleph_1^M, 2)$,

then $M[G] \models 2^{\aleph_0} = 2^{\aleph_1} = \kappa$.

We therefore need a different forcing to separate $2^{\aleph_0}$ and $2^{\aleph_1}$.

$Fn(X, Y, \lambda) := \{p : p$ is a partial fn from X to Y and $|p| < \lambda\}$

Note $Fn(X, Y) = Fn(X, Y, \aleph_0)$

If λ is, e.g., $\aleph_1^M$, then $Fn(X, Y, \aleph_1^M)$ does not have the c.c.c. anymore.

<u>Can show</u> [using a generalised version of Δ-SL]

$Fn(X, 2, \kappa^+)$ has the $(2^\kappa)^+$-c.c.

In particular $Fn(X, 2, \aleph_1^M)$ has the $(2^{\aleph_0})^+$-c.c.

Thus if $M \models CH$, it has the $(\aleph_1)^+$-c.c.

$= \aleph_2$-c.c.

Therefore, forcing with $Fn(X, 2, \aleph_1^M)$ will preserve cardinals $\geq \aleph_2^M$. The only cardinal we have to worry about is $\aleph_1^M$.

Def. A partial order is called $\aleph_1$-closed if for every countable descending chain [i.e., $\{ p_\alpha ; \alpha < \xi \}$ s.t. $\forall \alpha < \beta \quad p_\beta < p_\alpha$] there is some $q \in \mathbb{P}$ s.t. $q \leq p_\alpha$ for all p_α in the chain.

Clearly, our $Fn(X, 2)$ are not $\aleph_1$-closed. Also, clearly, our $Fn(X, 2, \aleph_1^M)$ are $\aleph_1$-closed [take the union of the chain].

Theorem If $M \models \mathbb{P}$ is $\aleph_1$-closed, then $\mathbb{P}$ does not add new subsets of ω.

Proof. We show this by proving that if $f : \omega \longrightarrow 2$ is in $M[G]$, then $f \in M$. Let τ be a name for f and find $p \Vdash \tau : \check{\omega} \longrightarrow \check{2}$ by forcing theorem.

Now recursively define

$p_{i+1} \leq p_i$ to be s.t.

$p_{i+1} \Vdash \tau(\check{i}) = \check{f}(i)$.

By $\aleph_1$-closure, find $q \leq p_i$ for all i.

In particular,

$q \Vdash \tau(\check{i}) = \check{f}(i)$ for all i.

Define in M,

$g(i) :=$ the unique $b \in \{0,1\}$ s.t. $q \Vdash \tau(\check{i}) = \check{b}$.

Then $g \in M$ and $g = f$, thus $f \in M$.

q.e.d.

Corollary (1) Forcing with $Fn(X, 2, \aleph_1^M)$ preserves $\aleph_1$. [Collapsing $\aleph_1$ requires adding a subset of $\mathbb{N}$.]

(2) Clearly if $\mathcal{P}(\mathbb{N})^M = \mathcal{P}(\mathbb{N})^{M[G]}$, then $(2^{\aleph_0})^{M[G]} = (2^{\aleph_0})^M$.

Final application

Start from a model

$$M \models 2^{\aleph_0} = \aleph_1 + 2^{\aleph_1} = \aleph_2$$

and force with

$$\mathbb{P} = Fn(\aleph_3^M \times \aleph_1^M, 2, \aleph_1^M),$$

then

$$M[G] \models 2^{\aleph_0} = \aleph_1 + 2^{\aleph_1} = \aleph_3.$$