XIII

THIRTEENTH & FINAL LECTURE

The Constructible Universe

8 July 2024

The projective hierarchy

Lecture XII p. 16

$$
\begin{array}{lcl}
\Sigma^1_3 \ \mathcal{PCPCA} & & \mathcal{CPCPCA} \ \Pi^1_3 \\
\quad\ \cup\!\!\!| & \subsetneq & \quad\ \cup\!\!\!| \\
\Sigma^1_2 \ \mathcal{PCA} & & \mathcal{CPCA} \ \Pi^1_2 \\
\quad\ \cup\!\!\!| & \subsetneq & \quad\ \cup\!\!\!| \\
\Sigma^1_1 \ \mathcal{P}\cancel{\mathcal{B}} = \mathcal{A} & & \mathcal{CA} \ \Pi^1_1 \\
& \cup\!\!\!| \quad \cup\!\!\!| & \\
& \mathcal{B} & \text{Borel}
\end{array}
$$

This is not a very convenient naming convention ⇝ Σ/Π.

Remark. Σ^1_n, Π^1_n is really equivalent to Σ_n, Π_n resp. for the right language.

$$\boxed{\begin{aligned} \Sigma^1_1 &:= \mathcal{A} \\ \Pi^1_n &:= \mathcal{C}\Sigma^1_n \\ \Sigma^1_{n+1} &:= \mathcal{P}\Pi^1_n \end{aligned}}$$

* We'll show that in $\mathbb{L}$, there's a Σ^1_2 set without p.s.p. (slightly weaker).

Preview for Lecture XIII: In $\mathbb{L}$, there is a Π^1_1 set without p.s.p *

Perfect set property: either countable or contains a nonempty perfect set

Set w/o PSP: uncountable set w/o nonempty perfect subset

Bernstein set: uncountable set X s.t. $\forall \alpha < 2^{\aleph_0}$

$$[T_\alpha] \cap X \neq \emptyset$$
$$[T_\alpha] \setminus X \neq \emptyset$$

Proved using AC: Bernstein sets exist. Cannot have PSP.

A few remarks

(1) The proof that $\mathbb{L}$ has a Π^1_1 set w/o p.s.p. requires slightly more DST than possible in 90 minutes.

[Technique very similar to those today.]

(2) What about

$$\Phi := \text{"every } \Sigma^1_2 \text{ set has p.s.p."}$$

Clearly, $\mathbb{L} \models \neg \Phi$.

With the "outer model technique"

SOLOVAY'S MODEL → (forcing), it is possible to produce such models, provided that you start from a model of ZFC + "there is an inaccessible cardinal".

(3) That additional assumption is necessary:

If $M \models \mathrm{ZFC} + \Phi$, then

$$\mathbb{L}^M \models \mathrm{ZFC} + \text{there is an inaccessible cardinal}$$

§ 26 Descriptive Complexity

Relationship of Σ^1_n classes and quantifiers.

Remember that projection: $\exists A$ was

$$A \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$$

$$\exists A = \{ y ; \exists x \ (x,y) \in A \}$$

So therefore if $A \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ is

Borel, Π^1_1, Π^1_2, Π^1_3, Π^1_4, ...

then $\{ x ; \exists y \ (x,y) \in A \}$ is

Σ^1_1, Σ^1_2, Σ^1_3, Σ^1_4, Σ^1_5 ...

The same with $\forall$: this is because $\forall = \neg \exists \neg$

if $A \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ is

Borel, Σ^1_1, Σ^1_2, Σ^1_3, Σ^1_4, ...

then $\{ x ; \forall y \ (x,y) \in A \}$ is

Π^1_1, Π^1_2, Π^1_3, Π^1_4, Π^1_5.

Quantifiers over $\mathbb{N}$ correspond to ctble unions & intersections, so if $\{ A_n ; n \in \mathbb{N} \}$ is a family of Borel sets, s.t. $\psi(x,k) \longleftrightarrow x \in A_k$, then

$$A = \{ x ; \exists k \ \psi(x,k) \} = \bigcup A_k \text{ is Borel.}$$

And the same for $\forall n \in \mathbb{N}$.

So, this is how we'll show that sets are Borel, Σ^1_1, Π^1_1, Σ^1_2 : produce the corresponding formulas.

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§ 27. Coding.

The elts of Cantor space are infinite binary sequences ; therefore the natural object that encodes a countable structures.

Let $(M, \in)$ be any countable infinite structure.

Since M is countable, there is

$$f : M \longrightarrow \mathbb{N} \text{ bijection.}$$

Define $E \subseteq \mathbb{N} \times \mathbb{N}$ by

$$n E m :\Longleftrightarrow f^{-1}(n) \in f^{-1}(m)$$

Then f is an isomorphism between $(M, \in)$ and $(\mathbb{N}, E)$.

Now take the Cantor bijection $b : \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}$

$c(E) \in 2^{\mathbb{N}}$ the code for E

$c(E)(n) = 1 \Longleftrightarrow b(n) = (k, \ell)$ & $k E \ell$

So $c(E)$ encodes $(M, \in)$ up to isomorphism

Suppose $g : \mathbb{N} \longrightarrow \overline{\mathbb{N}}$. That can be encoded by

$$c(g) := \underbrace{0 \dots 0}_{g(0) \text{ times}} 1 \underbrace{0 \dots 0}_{g(1) \text{ times}} 1 \underbrace{0 \dots 0}_{g(2) \text{ times}} 1 \dots \quad \text{etc.}$$

We call $x \in 2^{\mathbb{N}}$ a code if x has ∞ many 1s (i.e., $\forall k \exists \ell \; \ell > k \; \& \; x(\ell) = 1$)

Write $d(x)$ s.t. $c(d(x)) = x$. if x is a code

By our discussion of quantifiers, the set of codes is Borel.

What does it mean that $(\mathbb{N}, E)$ is wellfounded?

$$\forall g : \mathbb{N} \longrightarrow \mathbb{N} \; \exists n \; g(n+1) \not\mathrel{E} g(n)$$

$$\iff \underbrace{\forall x \in 2^{\mathbb{N}} \, (x \text{ is a code} \longrightarrow \underbrace{\exists n \neg \, d(x)(n+1) \mathrel{E} d(x)(n)}_{\text{Borel}})}_{\Pi^1_1}$$

Also : an assignment

$v_n \longmapsto$ an element of the structure

is represented by a function $a : \mathbb{N} \longrightarrow \mathbb{N}$,
and can be coded as well as an elt of $2^{\mathbb{N}}$.

<u>Proposition</u> Let φ be any fmla. Then

$$M_\varphi = \{ (c(E), c(a)) ; (\mathbb{N}, E, a) \models \varphi \}$$

is Borel.

<u>Proof.</u> Simple induction on complexity of φ.
For $\wedge, \vee, \neg$ it follows from closure of Borel sets under $\cap, \cup$, complement.
For $\exists, \forall$ it follows from closure of Borel sets under ctble unions & intersections. q.e.d.

<u>Corollary</u> $M_{Ext} := \{ c(E) ; (\mathbb{N}, E)$ is extensional $\}$

- is Borel.

Suppose $(\mathbb{N}, E)$ is wellfdd & extensional.

Let $\pi_E : \mathbb{N} \longrightarrow M$ be the Mostowski collapse.

<u>Proposition</u> $M_n := \{ (c(E), k) ;\ \pi_E(k) = n \}$ is Borel.

<u>Proof.</u> First observe that $M_0 = \{ (c(E), k) ;\ \forall \ell \ \neg \ell E k \}$ that's Borel.

Since $(c(E), k) \in M_{n+1} \iff \forall \ell \ (\ell E k \longleftrightarrow \bigvee_{i \leq n} ((c(E), \ell) \in M_i))$

By induction, all M_n are Borel.

q.e.d.

<u>Corollary</u> If $x \in 2^{\mathbb{N}}$

$M_x := \{ (c(E), k) ;\ \pi_E(k) = x \}$

is also Borel.

<u>Proof.</u> $\pi_E(k) = x \iff$

$$\forall n \quad x(n) = 0 \longleftrightarrow \exists i \exists j \ \pi_E(i) = n \wedge \pi_E(j) = 0 \wedge (i,j) E k$$

$$\wedge \ x(n) = 1 \longleftrightarrow \exists i \exists j \ \pi_E(i) = n \wedge \pi_E(j) = 1 \wedge (i,j) E k. \quad \text{qed.}$$

§ 28 Descriptive complexity in $\mathbb{L}$.

Warm-up We showed that there is a definable well-order $<_{\mathbb{L}}$ in $\mathbb{L}$.

What is the descriptive complexity of

$$<_{\mathbb{L}} \cap\, 2^{\mathbb{N}} \times 2^{\mathbb{N}} \,?$$

Let ψ be the formula defining $<_{\mathbb{L}}$:

i.e., $x <_{\mathbb{L}} y \iff \mathbb{L} \models \psi(x,y)$.

Remember that we need that ψ is absolute for transitive models.

By earlier techniques, we find $\alpha < \omega_1$ s.t. $\mathbb{L}_\alpha \models \psi(x,y)$.

As a consequence (MINIATURISATION)

$$x <_{\mathbb{L}} y \iff \exists \alpha < \omega_1 \;\; \mathbb{L}_\alpha \models \psi(x,y)$$

$$\iff \exists E \, \exists k \, \exists \ell \;\; (\mathbb{N}, E)$$

$\Pi^1_1 \longrightarrow$ wellfdd & extensional

Borel $\longrightarrow$ & $\pi_E(k) = x$

Borel $\longrightarrow$ & $\pi_E(\ell) = y$

Borel $\longrightarrow$ & $(\mathbb{N}, E) \models \psi(k, \ell)$

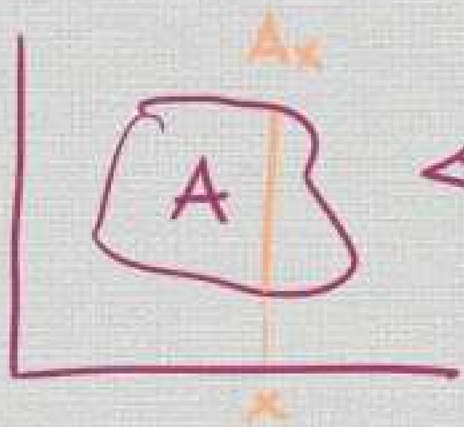
Borel $\longrightarrow$ & $(\mathbb{N}, E) \models \sigma^*$ THE CONDENSATION SENTENCE

So, we have proved that

$$<_{\mathbb{L}} \cap\, 2^{\mathbb{N}} \times 2^{\mathbb{N}}$$

is a Σ^1_2 set.

Remark If $< \cap\, 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ is a wellorder of order type ω_1, then Fubini's theorem implies that it is not Lebesgue measurable.



$$\mu(A) = 0 \iff \mu(\{x;\ \mu(A_x) > 0\}) = 0$$

Thus: in $\mathbb{L}$, there is a Σ^1_2 set that is not Lebesgue measurable.

Reconstruction of the proof of Bernstein's thm.

Lecture XII

Proof of Bernstein

A set B is called Bernstein set if for every $T \in$ PfTree we have
$$[T] \cap B \neq \emptyset$$
&
$$[T] \cap 2^{\mathbb{N}} \setminus B \neq \emptyset$$

Note if B is an uncountable Bernstein set, B does not have p.s.p.

We'll construct such a Bernstein set.

Construct B_α, A_α in stages s.t.
$$B_\alpha \cap A_\alpha = \emptyset$$
s.t. there is a_α, b_α s.t.
$$a_\alpha \in [T_\alpha] \cap A_\alpha$$
$$b_\alpha \in [T_\alpha] \cap B_\alpha .$$

Then let $B := \bigcup_{\alpha \in 2^{\aleph_0}} B_\alpha$. This by construction is an uncountable (size $2^{\aleph_0}$) Bernstein set.

Do this recursively:

Fix α and suppose $\forall \xi < \alpha$ A_ξ, B_ξ are defined
$$A^* := \bigcup_{\xi<\alpha} A_\xi \quad B^* = \bigcup_{\xi<\alpha} B_\xi$$
with $|A^*| = |\alpha| = |B^*|$.

Then since by Cantor's Lemma $|[T_\alpha]| = 2^{\aleph_0}$. and thus
$$|[T_\alpha] \setminus \underbrace{(A^* \cup B^*)}_{\substack{\text{small} \\ |\alpha| \cdot 2}}| = 2^{\aleph_0}$$

Use AC to pick two of these, call them a_α, b_α.
$$A_\alpha := A^* \cup \{a_\alpha\} \subseteq 2^{\mathbb{N}}$$
$$B_\alpha := B^* \cup \{b_\alpha\}.$$

All induction assumptions are satisfied.

q.e.d.

Def. A sequence $S = ((b_\alpha, a_\alpha, T_\alpha) ; \alpha < \gamma)$ for $\gamma < \omega_1$ is called a Bernstein sequence of length γ if

1. T_α is the $<_L$-least perfect tree not in $\{T_\beta ; \beta < \alpha\}$.
2. b_α is the $<_L$-least elt of $[T_\alpha] \setminus (\{b_\beta ; \beta < \alpha\} \cup \{a_\beta ; \beta < \alpha\})$
3. a_α is the $<_L$-least elt of $[T_\alpha] \setminus (\{b_\beta ; \beta < \alpha\} \cup \{a_\beta ; \beta < \alpha\})$

We say T occurs in S if it's one of the T_α $(\alpha<\gamma)$;

we say $x\in 2^{\mathbb{N}}$ occurs in S if it's one of the a_α or b_α $(\alpha<\gamma)$.

$$x\in B \iff \underbrace{\begin{array}{l}\exists S \text{ Bernstein seq. and} \\ \exists T\ T \text{ is perfect and } <_{\mathbb{L}}\text{-least not occurring in } S \text{ and} \\ x \text{ is } <_{\mathbb{L}}\text{-least not occurring in } S \\ \quad\text{elt of } [T]\end{array}}_{\Phi(x)}$$

Note that if ϑ is big enough s.t. $S,T,x\in\mathbb{L}_\vartheta$, then $\mathbb{L}_\vartheta\models\Phi(x) \implies \mathbb{L}\models\Phi(x)$.

Again with our previous miniaturisation technique, get $\alpha<\omega_1$ s.t.

$$x\in B \iff L_\alpha\models\Phi(x).$$

$$\iff \left\{\begin{array}{l}\underset{\exists k}{\underline{\exists E}}\ \overbrace{(\mathbb{N},E) \text{ is wellfdd and ext.}}^{\Pi^1_1}\ \&\ \pi_E(k)=x \\ \&\ (\mathbb{N},E)\models\Phi(k) \\ \&\ (\mathbb{N},E)\models\sigma^*\end{array}\right. \quad \text{q.e.d.}$$

Σ^1_2

Borel

THE CONDENSATION SENTENCE