XII

Lecture XII
Constructible Universe

1 July 2024

Recap Last time we introduced

PSP : A has the p.s.p. iff
A is either countable or
A contains a non-empty
pf. subset

$\Longrightarrow |A| \geq 2^{\aleph_0}$

Seen : PSP (All sets have p.s.p.) $\Longrightarrow$ CH.

However AC $\longrightarrow \neg$PSP.

Today: Definitions & Theorems.

"Descriptive Set Theory"

— Relationship between properties of sets (of reals) and their descriptions.

[$\rightsquigarrow$ LOGIC]

§ 25 Perfect sets & Bernstein's theorem

We will deal with Cantor space

$$2^{\mathbb{N}} := \{ f \,;\, f : \mathbb{N} \longrightarrow 2 \}$$

This is almost the same as $\mathcal{P}(\mathbb{N})$ via the bijection

$$A \longmapsto \chi_A$$

[the characteristic fn of A
$\chi_A(n) = 1 \Leftrightarrow n \in A$]

So obviously

$$|2^{\mathbb{N}}| = 2^{\aleph_0}.$$

This is in bijection with the reals $\mathbb{R}$.

Can think of $f \in 2^{\mathbb{N}}$ as

$$\sum_{n \in \mathbb{N}} f(n) 2^{-(n+1)} \in \mathbb{R}$$
$$\in [0,1]$$

But that's not quite a bijection since

010000 ....
001111 ......

give the same real number.

Define a distance fn on Cantor space; $x, y \in 2^{\mathbb{N}}$

$$d(x,y) = \begin{cases} 0 & x = y \\ 2^{-n} & \text{where } n \text{ is minimal} \\ & \text{s.t. } x(n) \neq y(n) \end{cases}$$

This gives rise to a metric space, therefore a topological space. This is homeomorphic to the Cantor 1/3-set.

This gives rise to the standard metric space theory: open / closed

$A \subseteq 2^{\mathbb{N}}$ closed $\iff$ $\forall \{x_n; n \in \omega\} \subseteq A$ if $x_n \longrightarrow x$, then $x \in A$.

Also $\{x\} \subseteq 2^{\mathbb{N}}$ is closed.

In particular, all countable sets are countable unions of closed sets.

F_σ

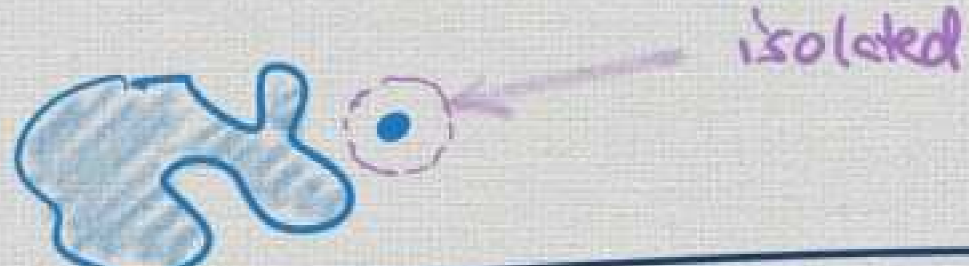
Fermé

Def. $A \subseteq 2^{\mathbb{N}}$ is called perfect if it's closed and has no isolated points

i.e., x is isolated if there is $\varepsilon > 0$ s.t. for all $y \in A$

$$d(x,y) < \varepsilon \longrightarrow y = x.$$

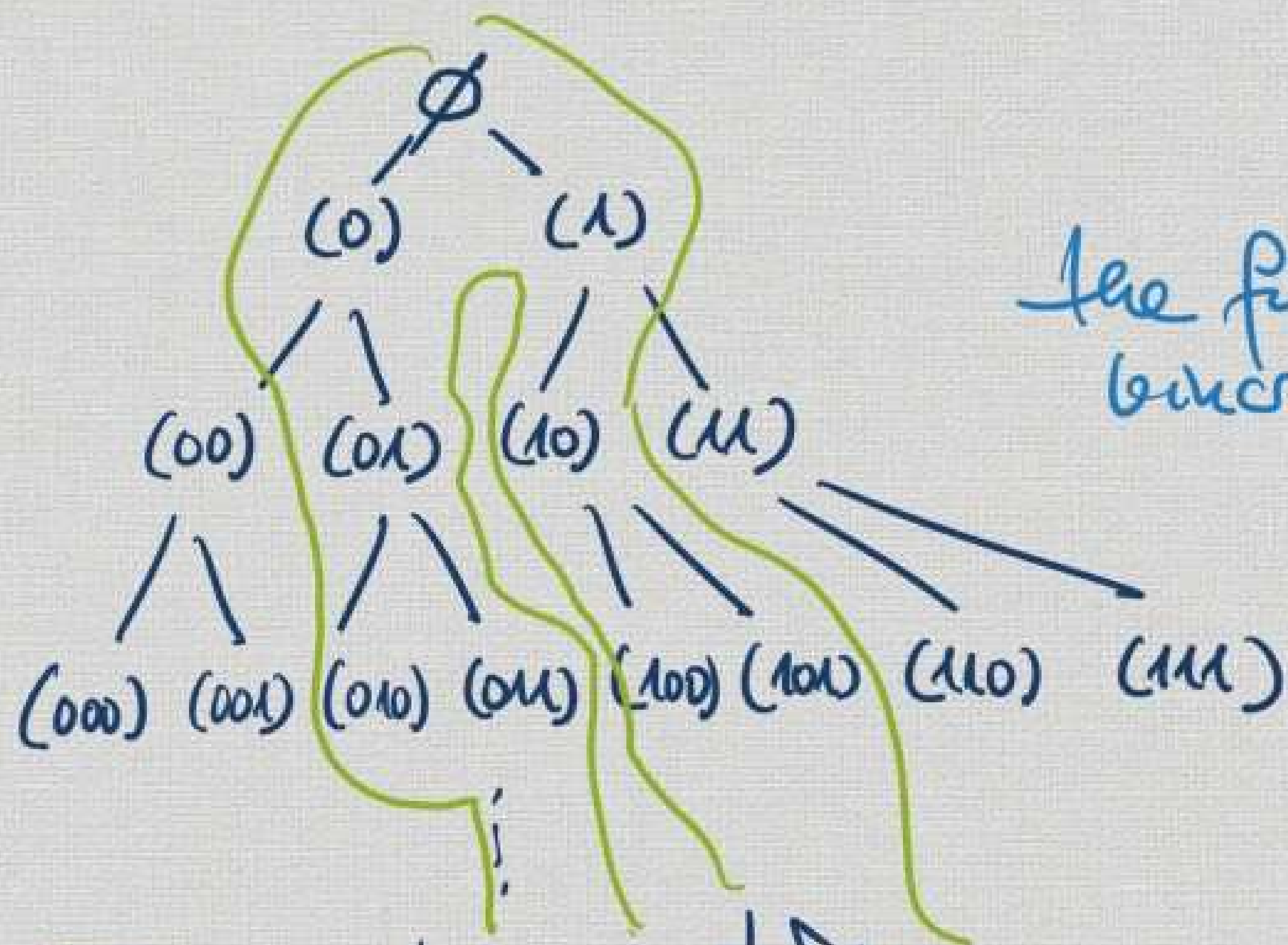
Picture



In Cantor space, we can characterize being closed & being perfect by combinatorial objects: trees.

Def. A subset $T \subseteq 2^{<\mathbb{N}} = \bigcup_{n\in\mathbb{N}} 2^n$ is called a tree if it's closed under initial segments:

if $t \in T$ then for all $s \subseteq t$, $s \in T$



the full binary tree.

If T is a tree, define

$$[T] := \{ x \in 2^{\mathbb{N}} ; \forall n \ x \restriction n \in T \}$$

the set of branches through T

If $A \subseteq 2^{\mathbb{N}}$, define

$$T_A := \{ x \restriction n \ ; \ x \in A \ \& \ n \in \mathbb{N} \}$$

the tree defined by A.

By definition, $A \subseteq [T_A]$ for all A.

Converse $[T_A] \subseteq A$ not true in general:

$$A := \{x = 0^k 1 y \,;\, k \in \mathbb{N}, y \in 2^{\mathbb{N}}\}$$

$\vec{0} \notin A$, but $0^k \in T_A$ for all k,

so $\vec{0} \in [T_A]$.

<u>Proposition</u> A is closed $\iff A = [T_A]$.

<u>Proof.</u> Only need to show $[T_A] \subseteq A$.

"$\Longrightarrow$" If A closed and $x \in [T_A]$, then for all n $x\upharpoonright n \in T_A$, so there is some $x_n \in A$ s.t. $x_n \upharpoonright n = x \upharpoonright n$, so

$$d(x_n, x) \leq 2^{-n}.$$

So $x_n \longrightarrow x$, so by closure, $x \in A$.

"$\Longleftarrow$". Assume that $x_n \in A$ and $x_n \to x$.

Need to show $x \in A$.

$\forall k \exists n$

$x_n \upharpoonright k = x \upharpoonright k$

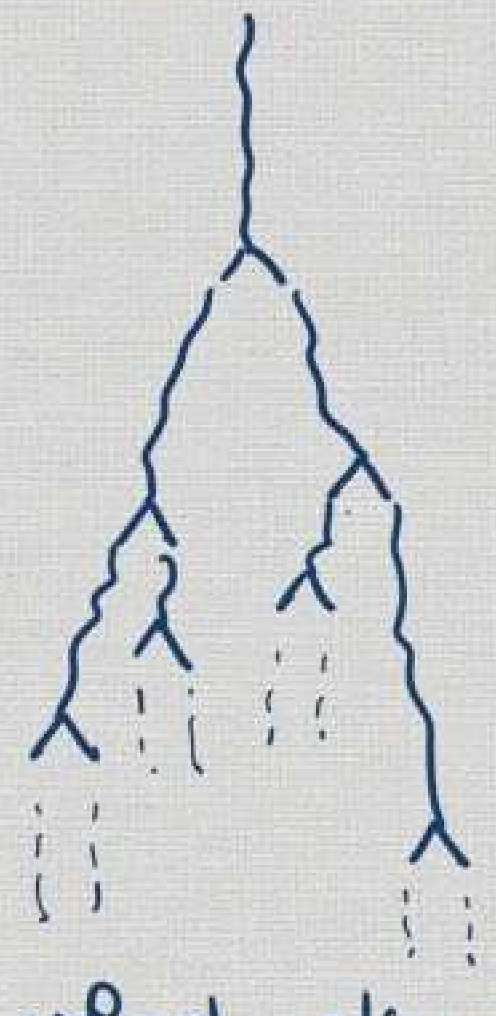
$\Longleftarrow$

$\forall k$ $x\upharpoonright k \in T_A$

$\Longrightarrow x \in [T_A] = A$. q.e.d.

Def. If T is a tree, $t \in T$ is called a splitting node if $t0, t1 \in T$.

T is called perfect if for $t \in T$ there is $s \supseteq t$, $s \in T$, s is a splitting node of T.



perfect tree

Clearly, if T is perfect, then

$$|[T]| = 2^{\aleph_0}$$

[Every elt of $2^{\mathbb{N}}$ gives infinitely many left-right decisions; this defines inj. from $2^{\mathbb{N}}$ into $[T]$.]

Proposition $A \subseteq 2^{\mathbb{N}}$ is perfect (i.e., closed w/o isolated pts)
$A \neq \emptyset$.

$\iff T_A$ is perfect (as a tree).

Proof. If A is perfect, then for each x

"$\Rightarrow$" and every ε, I find $y \neq x$ s.t. $d(x,y) < \varepsilon$.

This means that there is some k s.t.

$$x \upharpoonright k = y \upharpoonright k$$
$$x(k) \neq y(k)$$

so this is a splitting node in T_A

and k is large enough that $2^{-k} < \varepsilon$.

"$\Leftarrow$". If $x \in A$ is isolated, then take ε s.t. no element of A other than x is within ε of x.

Find k s.t. $2^{-k} < \varepsilon$, then there are no splitting nodes of T_A beyond $x \upharpoonright k$.

q.e.d.

Corollary (Cantor's Lemma)
If $A \neq \emptyset$ perfect, then $|A| = 2^{\aleph_0}$.

Def. A has the p.s.p. iff
A is either countable or contains a perfect subset.

Thus sets with p.s.p. cannot be counterexamples to CH.

Idea Prove CH by proving that every set has p.s.p.

That didn't quite work out, as we'll see.

Thm (Cantor-Bendixson) Without proof
Every closed set has p.s.p.

Thm (Hausdorff) Without proof
Every Borel set has p.s.p.

Theorem (Bernstein)

$AC \Longrightarrow$ there are sets w/o p.s.p.

Let's first count how many perfect trees we have:

$T \subseteq 2^{<\mathbb{N}}$ Note $2^{<\mathbb{N}}$ is countable.

Therefore $\{T \subseteq 2^{<\mathbb{N}} ; T \text{ is tree}\} =: \text{Tree}$ injects in $2^{\mathbb{N}}$, so

$|\text{PfTree}| \leq |\text{Tree}| \leq 2^{\aleph_0}$.

If $X \subseteq \mathbb{N}$ is co-infinite, let T_X be the full binary tree with all splitting node of length $n \in X$ removed. Because X is co-infinite T_X is perfect.

Inj. $X \longmapsto T_X$ from co-infinite subsets of $\mathbb{N}$ to perfect trees.

$2^{\aleph_0} \leq |\text{PfTree}|$

set of pf trees

So, I can write

$\{T_\alpha ; \alpha < 2^{\aleph_0}\}$ pf trees listed in order type $2^{\aleph_0}$.

Proof of Bernstein

A set B is called Bernstein set if for every $T \in \mathrm{PfTree}$ we have

$$[T] \cap B \neq \emptyset$$
$$\& \quad [T] \cap 2^{\mathbb{N}} \setminus B \neq \emptyset$$

Note if B is an uncountable Bernstein set, B does not have p.s.p.

We'll construct such a Bernstein set.

Construct B_α, A_α in stages s.t.

$$B_\alpha \cap A_\alpha = \emptyset$$

s.t. there is a_α, b_α s.t.

$$a_\alpha \in [T_\alpha] \cap A_\alpha$$
$$b_\alpha \in [T_\alpha] \cap B_\alpha.$$

Then let $B := \bigcup_{\alpha \in 2^{\aleph_0}} B_\alpha$. This by construction is an uncountable (size $2^{\aleph_0}$) Bernstein set.

Do this recursively:

Fix α and suppose $\forall \xi < \alpha$

A_ξ, B_ξ are defined

$$A^* := \bigcup_{\xi<\alpha} A_\xi \qquad B^* = \bigcup_{\xi<\alpha} B_\xi$$

with $|A^*| = |\alpha| = |B^*|$.

Then since by Cantor's Lemma

$|[T_\alpha]| = 2^{\aleph_0}$. and thus

$$\Big| [T_\alpha] \setminus \underbrace{(A^* \cup B^*)}_{\substack{\text{small} \\ |\alpha|+2}} \Big| = 2^{\aleph_0}$$

Use AC to pick two of these,

call them a_α, b_α. $\in 2^{\mathbb{N}}$

$$A_\alpha := A^* \cup \{a_\alpha\}$$

$$B_\alpha := B^* \cup \{b_\alpha\}.$$

All induction assumptions are satisfied.

q.e.d.

Remark In $\mathbb{L}$, we have a definable wellordering that might allow us to specify how complex that Bernstein set is in $\mathbb{L}$.

§ 26 Descriptive Complexity

Borel sets: If X is any topological space, that's the smallest σ-algebra containing the closed sets.

closed under complement & countable union

Lebesgue's error

FALSE

If $f: 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is continuous & $B \subseteq 2^{\mathbb{N}}$ is Borel, then ~~$f[B]$ is Borel~~.

not true: Suslin

Suslin's idea: consider a third operation projection, close under this and obtain a new hierarchy:

the projective hierarchy.

Projection? Note that $\mathbb{R} \not\cong \mathbb{R}^2$.

$2^{\mathbb{N}} \times 2^{\mathbb{N}}$ is homeomorphic to $2^{\mathbb{N}}$ by interlacing:

$x \in 2^{\mathbb{N}}$ $\quad (x)_{I}(n) := x(2n)$

$\quad (x)_{II}(n) := x(2n+1)$

$x, y \in 2^{\mathbb{N}}$ $\quad x * y := (x(0), y(0), x(1), y(1), x(2), y(2), \ldots$

Then: $(x)_{I} * (x)_{II} = x$

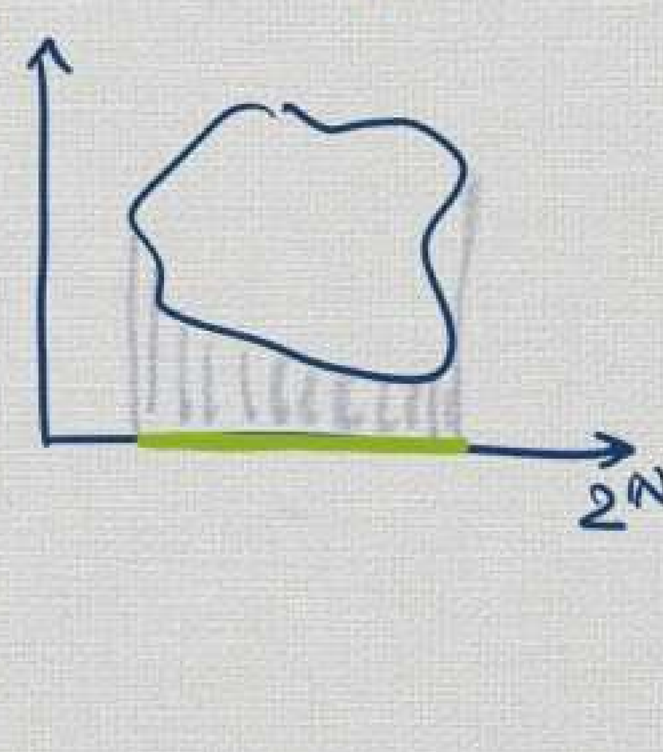
$(x*y)_{I} = x \quad (x*y)_{II} = y.$

Similarly $(2^{\mathbb{N}})^k \cong 2^{\mathbb{N}}$.

We call Cantor space zero-dimensional.

If $A \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$, what is the projection of A?

$\exists A := \{ y ; \exists x \ (x,y) \in A \}$



We call a $\Gamma \subseteq \mathcal{P}(2^{\mathbb{N}})$ a pointclass.
E.g., closed, open, Borel, F_σ

Def $C\Gamma := \{ 2^{\mathbb{N}} \setminus A ; A \in \Gamma \}$
known as the dual pointclass

forming the complement

$P\Gamma := \{ \exists A ; A \in \Gamma^2 \}$
where $\Gamma^2 = \{ A \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}},$ A is in Γ via the canonical homeomorphism$\}$

Suslin defined the analytic sets

$\mathfrak{A} := P\mathcal{B}$ where $\mathcal{B}$ is the Borel sets

By Lebesgue's error, $\mathfrak{A} \supsetneqq \mathcal{B}$.
[Get $\mathfrak{A} \supseteq \mathcal{B}$ by considering the cylinder $C_B := \{ (x,y) ; y \in B \}$ for Borel sets B. Clearly $\exists C_B = B$.]

Suslin also showed that

$$\mathfrak{A} \cap C\mathfrak{A} = \mathfrak{B},$$

so in particular $C\mathfrak{A} \neq \mathfrak{A}$.

The projective hierarchy:

$$\begin{array}{ccccc}
 & & \vdots & & \\
\Sigma^1_3 & PCPC\mathfrak{A} & & CPCPC\mathfrak{A} & \Pi^1_3 \\
 & \rotatebox[origin=c]{90}{$\subseteq$} & \subseteq & \rotatebox[origin=c]{90}{$\subseteq$} & \\
\Sigma^1_2 & PC\mathfrak{A} & & CPC\mathfrak{A} & \Pi^1_2 \\
 & \rotatebox[origin=c]{90}{$\subseteq$} & \subseteq & \rotatebox[origin=c]{90}{$\subseteq$} & \\
\Sigma^1_1 & P\mathfrak{B} = \mathfrak{A} & & C\mathfrak{A} & \Pi^1_1 \\
 & \subseteq & & \subseteq & \\
 & & \mathfrak{B} & &
\end{array}$$

This is not a very convenient naming convention $\rightsquigarrow$ Σ/Π.

Remark. Σ^1_n, Π^1_n is really equivalent to Σ_n, Π_n resp. for the right language.

$$\boxed{\begin{aligned} \Sigma^1_1 &:= \mathfrak{A} \\ \Pi^1_n &:= C\Sigma^1_n \\ \Sigma^1_{n+1} &:= P\Pi^1_n \end{aligned}}$$

Preview for Lecture XIII

In $\mathbb{L}$, there is a Π^1_1 set without p.s.p