XI THE CONSTRUCTIBLE UNIVERSE

Eleventh Lecture

24. Juni 2024

DIAMOND PRINCIPLE

Lecture V

Stationary sets are still somewhat big.

Definition Let $(A_\alpha ; \alpha < \omega_1)$ be a sequence of sets s.t. $\boxed{A_\alpha \subseteq \alpha} < \omega_1$.
Then $(A_\alpha ; \alpha < \omega_1)$ called a DIAMOND SEQUENCE if for all $A \subseteq \omega_1$, the set

$$A^* := \{\alpha < \omega_1 ; \boxed{A \cap \alpha = A_\alpha}\}$$

is stationary.

A is STATIONARY if $\forall C$ club $A \cap C \neq \emptyset$

C is CLUB if it is closed & unbounded

A diamond seq. is a fixed seq. that predicts the initial segments of any set $A \subseteq \omega_1$ on a "somewhat big" set.

Jensen: $\Diamond \Longrightarrow \neg SH$.

Theorem (Jensen 1972)

$$\mathbb{L} \models \Diamond$$

We are in the middle of that proof.

Define

$\Psi(\alpha, X, Y, s) :\iff$ α is a limit ordinal, $X \subseteq \alpha$, $Y \subseteq \alpha$, Y is a club set, $dom(s) = \alpha$ and $\forall \eta \in Y \; X \cap \eta \neq s(\eta)$.

Note that Ψ is absolute for transitive models.

[Ψ says that Y witnesses that X hasn't been guessed correctly so far.]

Y takes the role of C

X takes the role of A

"$A^* \cap C = \emptyset$".

We define functions S (and auxiliary fns A and C) by recursion on ω_1:

$S(\alpha) = (\emptyset, \emptyset)$ for any α successor or 0

$S(\alpha) = (X, Y) \rightsquigarrow A(\alpha) := X$, $C(\alpha) := Y$

If α is limit:

If there is no pair (X, Y) s.t. $\Psi(\alpha, X, Y, A\restriction\alpha)$ holds

then $S(\alpha) := (\emptyset, \emptyset)$

o/w let X, Y be the $\underline{L}$-least pair s.t. $\Psi(\alpha, X, Y, A\restriction\alpha)$ holds.

uses $V=L$

$\Sigma(\alpha, X, Y)$ for $S(\alpha) = (X, Y)$

We claim that $(A(\alpha); \alpha < \omega_1)$ is a diamond sequence.

$(A(\alpha), C(\alpha))$ is the $\mathbb{L}$-least pair s.t.

$\Phi(\alpha, A(\alpha), C(\alpha), A\upharpoonright\alpha)$ holds

[if such a pair exists].

Suppose towards a contradiction that it is not a diamond sequence, so there is $E \subseteq \omega_1$ and D club s.t.

$$D \cap E^* = D \cap \{\alpha ; E \cap \alpha = A(\alpha)\} = \emptyset.$$

$$\Longleftrightarrow$$

$$\Phi(\omega_1, E, D, A)$$

Let (D, E) be the $\mathbb{L}$-least pair s.t. $\Phi(\omega_1, E, D, A)$

We observed $\omega_1, D, E, A, S \in \mathbb{H}_{\omega_2}$.

Then:

(*) $\mathbb{H}_{\omega_2} \models (D,E)$ is the $\mathbb{L}$-least pair s.t. $\Phi(\omega_1, D, E, A)$.

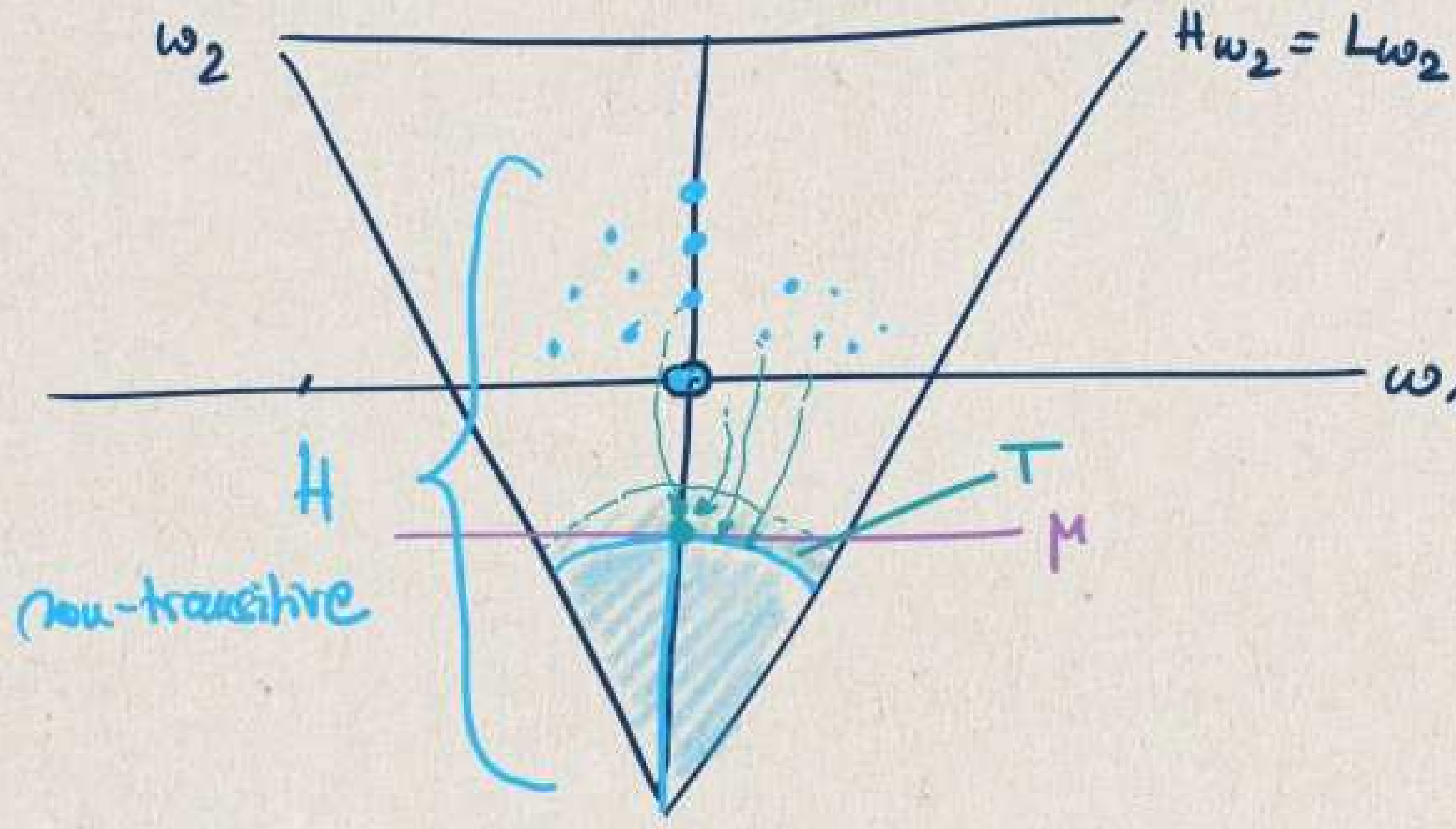
Let $H \prec \mathbb{H}_{\omega_2}$ be countable with $\omega_1, D, E, A, S \in H$.

Tarski-Vaught + unique definability of ω_1, D, E, A, S in $\mathbb{H}_{\omega_2}$.

We proved:

Claim 1 $H \cap \omega_1 =: \mu$ is a countable ordinal.

$\boxed{\text{END OF RECAP}}$



$H \prec H_{\omega_2}$

transitive

Consider the Mostowski collapse $\pi : H \longrightarrow T$

$$\pi(x) := \{\pi(y) ;\ y \in x \cap H\}$$

recursive definition

Since μ is transitive, $\mu \subseteq H$, we have

$$\pi \upharpoonright \mu = id \upharpoonright \mu .$$

Consider the inverse of π:

$$\pi^{-1} : T \longrightarrow H \xrightarrow{id} H_{\omega_2}$$

elementary since isomorphism

elementary embedding since $H \prec H_{\omega_2}$

<u>So:</u> $\pi^{-1} : T \longrightarrow H_{\omega_2}$ is an elementary embedding.

Therefore:

(*) $\mathbb{H}_{\omega_2} \models (D,E)$ is the $\mathbb{R}$-least s.t. $\Phi(\omega_1, D, E, A)$

$\Longrightarrow \quad T \models (\pi(D), \pi(E))$ is the $\mathbb{R}$-least s.t. $\Phi(\pi(\omega_1), \pi(D), \pi(E), \pi(A))$

Goal: Determine what all of these $\pi(x)$ are!

Calculations

(i) $\pi(\omega_1) = \mu$ since $\pi \restriction \mu = id \restriction \mu$ & $H \cap \omega_1 = \mu$. & $\omega_1 \in H$.

(ii) $\underline{\pi(E)} \underset{\text{Def of Mostowski}}{=} \{\pi(x);\ x \in E \cap H\}$

$E \subseteq \omega_1$

$= \{\pi(x);\ x \in E \cap \mu\}$

$\underset{\pi \restriction \mu = id \restriction \mu}{=} \{x;\ x \in E \cap \mu\} = \underline{E \cap \mu}.$

(iii) $\pi(D) = D \cap \mu$

[Same proof; replacing E by D.]

(iv) $\pi(A) \underset{\text{Claim}}{=\!=} A \restriction \mu$.

[$H_{\omega_2} \models A$ is a sequence of length ω_1

$\Longrightarrow T \models \pi(A)$ is a sequence of length $\pi(\omega_1) \underset{(i)}{=} \mu$.

$+ \pi \restriction \mu = \mathrm{id} \restriction \mu$

Fix $\gamma < \mu$.

$H_{\omega_2} \models A(\gamma)$ is the γth elt of A

$\Longrightarrow T \models \pi(A(\gamma))$ is the $\underbrace{\pi(\gamma)}_{=\gamma}$th elt of $\pi(A)$

This shows that

$$\pi(A) = (\pi(A(\gamma)); \gamma < \mu).$$

$H_{\omega_2} \models$ there is Y s.t. $\Sigma(\gamma, A(\gamma), Y)$

$\Longrightarrow T \models$ there is Y s.t. $\Sigma(\gamma, \pi(A(\gamma)), Y)$.

But Σ is absolute for transitive models, so the only set X that can have the property $T \models \exists Y\, \Sigma(\gamma, X, Y)$ is the set $A(\gamma)$. Thus $\pi(A(\gamma)) = A(\gamma)$.

$$\pi(A) = (A(\gamma); \gamma < \mu). \;]$$

We had:

$$T \models (\pi(D), \pi(E)) \text{ is the } \mathbb{L}\text{-least s.t. } \Phi(\pi(\omega_1), \pi(D), \pi(E), \pi(A))$$

$$\xRightarrow{(i)-(iv)} T \models (D\cap\mu, E\cap\mu) \text{ is the } \mathbb{L}\text{-least s.t. } \Phi(\mu, D\cap\mu, E\cap\mu, A\upharpoonright\mu).$$

By definition of S or Σ, we have

$$T \models \Sigma(\mu, D\cap\mu, E\cap\mu)$$

$$\Longrightarrow \quad \begin{array}{l} D\cap\mu = C(\mu) \\ E\cap\mu = A(\mu) \end{array} \Longrightarrow \mu \in E^*$$

Since we assumed towards a contradiction that $D\cap E^* = \emptyset$, it's enough to show $\mu \in D$.

<u>Claim 2</u> $\mu \in D$.

[$H_{\omega_2} \models D$ is unbounded in ω_1

$\Longrightarrow T \models \pi(D)$ is unbounded in $\pi(\omega_1)$

$\Longrightarrow T \models D\cap\mu$ is unbounded in μ.

$\rightarrow$ μ is a limit point of D

Since D is closed, we get $\mu \in D$.]

Thus we have obtained a contradiction.

q.e.d.

§23 Cantor's Theorem on dense total orders

<u>Theorem</u> (Cantor)
If $(D,\leq)$ is a countable, dense, unbounded total order, then $(D,\leq) \cong (\mathbb{Q},\leq)$.

<u>Proof.</u> Since $D, \mathbb{Q}$ are countable, enumerate them by $\mathbb{N}$, i.e.,

$$D = \{d_n ; n\in \mathbb{N}\}$$
$$\mathbb{Q} = \{q_n ; n\in \mathbb{N}\}$$

Write $D_i := \{d_k ; k<i\}$ $D_0 = \emptyset$.

Define by recursion a map
$$f : D \longrightarrow \mathbb{Q} \quad \text{s.t.}$$
$$f\restriction D_i : D_i \longrightarrow f[D_i]$$
is an isomorphism.

Define $f(d_0) := q_0$. Suppose that $f\restriction D_i$ is such an isomorphism.

What is the relationship between d_i and $D_i = \{d_k ; k<i\}$?

If $d_{k_0} < d_{k_1} < \ldots < d_{k_{i-1}}$

with $\{d_{k_j} ; j<i\} = \{d_k ; k<i\}$,

then d_i can either be

(1) below d_{k_0}

(2) above $d_{k_{i-1}}$

(3) between d_{k_j} & $d_{k_{j+1}}$ for unique j.

Similarly, we have

$$f(d_{k_0}) < f(d_{k_1}) < \ldots < f(d_{k_{i-1}})$$

in $\mathbb{Q}$. For each case (1), (2), or (3) there is a rational number $q \in \mathbb{Q}$ that has precisely the same position w.r.t. $f[D_i]$ as q_i has to D_i.

Let $f(d_i) = q_k$ where k is least among those.

Clear from construction that

$$f \upharpoonright D_{i+1}$$

is an isomorphism onto $f[D_{i+1}]$.

Claim $f : D \longrightarrow \mathbb{Q}$

is injective & order preserving.

[Obvious: if, e.g., $f(d_n) = f(d_m)$, let $i := \max(n, m) + 1$, then $f \upharpoonright D_i$ is not an iso. Contradiction.]

Remark. So far, we only used the countability of D.

So — COROLLARY.

Every countable total order is isomorphic to a subset of $\mathbb{Q}$.

[Universality of $\mathbb{Q}$ for countable total orders!]

Still to show: $f[D] = \mathbb{Q}$.

Suppose not, so there is q_n s.t. $q_n \notin \operatorname{ran}(f)$.

Let n be minimal with this property.

So $q_0, \dots, q_{n-1} \in \operatorname{ran}(f)$. Find k s.t.

$q_0, \dots, q_{n-1} \in \operatorname{ran}(f \upharpoonright D_k)$.

Clearly $\operatorname{ran}(f \upharpoonright D_k)$ is a finite subset of $\mathbb{Q}$. [exactly k elements]. So as before q_n is in position (1), (2) or (3) w.r.t. $\operatorname{ran}(f \upharpoonright D_k)$.

Using that $(D, \leq)$ is dense & unbounded, we get that there is some $d \in D$ that is precise in the same position w.r.t. D_k as q_n is w.r.t. $\operatorname{ran}(f \upharpoonright D_k)$.

Say $d = d_\ell$. Note $\ell > k$.

But then in step ℓ of the construction, we must have picked the smallest index m s.t. q_m is in the same position.

But $m = n$, so q_n would be assigned to d_ℓ. Contradiction!

q.e.d.

"BACK-AND-FORTH"

Slightly alternative way of writing this down! In the recursion do $f(d_i)$ in step $2i$ and find $f^{-1}(q_i)$ in step $2i+1$, making sure that no inconsistency appears. That is possible by the fact that $(D, \leq)$ is dense & unbounded & gets a surjection automatically.

§24 Effective versions of CH.

When Cantor tried to prove CH, he tried to embed "large sets", i.e., sets of cardinality $|\mathbb{R}|$ into arbitrary uncountable sets.

$$\text{CH} \iff \forall A \subseteq \mathbb{R} \left(|A| \le \aleph_0 \text{ or } A \sim \mathbb{R} \right)$$

An example is the <u>Cantor set</u>:

$C_0 := [0,1]$

$C_1 := [0,1/3] \cup [2/3,1]$

$C_2 := [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1]$

etc.

$C := \bigcap_{n\in\mathbb{N}} C_n$.

This set has card. $|\mathbb{R}| = 2^{\aleph_0}$. This is an example of <u>perfect sets</u> (defined in §25).

<u>Cantor proved:</u> $A \subseteq \mathbb{R}$ perfect $\Longrightarrow |A| = 2^{\aleph_0}$.

<u>Strengthening</u> of CH:

$$\forall A \subseteq \mathbb{R} \left(|A| \le \aleph_0 \text{ or } A \text{ contains a perfect subset} \right)$$

This is called

THE PERFECT (SUB)SET PROPERTY. PSP.

Thus: PSP $\Longrightarrow$ CH.

<u>Theorem</u> (Cantor & Bendixson).

If A is closed, then A has the perfect set property, i.e., $|A| \leq \aleph_0$ or A contains a perfect subset.

<u>Theorem</u> (Hausdorff).

If A is Borel, then A has the perfect set property.

<u>Q.</u> Can you extend this to all sets A?

A. NO!

$\mathrm{ZFC} \vdash \neg \mathrm{PSP}$.

[This proof needs the axiom of choice.]

<u>Natural question</u>: How far can you go?

Mikhail Y. Suslin

Born 15 November 1894
Krasavka, Saratov Oblast

Died 21 October 1919 (aged 24)
Krasavka, Saratov Oblast

Lebesgue's Error

Lebesgue assumed that the Borel sets are closed under continuous images, i.e., if

$f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous

$A \subseteq \mathbb{R}$ is Borel

then $\boxed{f[A] \text{ is Borel}}$

FALSE in general

Suslin observed that

$$\mathfrak{A} := \{ f[A] ; A \text{ Borel}, f \text{ cts} \}$$

ANALYTIC SETS

has the property that

$$\mathfrak{A} \supsetneqq \underbrace{\mathcal{B}}_{\text{Borel sets.}}$$

<u>Theorem (Suslin)</u> If A is analytic, then A has the perfect set property.

The complements of analytic sets are called <u>coanalytic</u>; coanalytic sets are not closed under images.

This leads to the projective hierarchy:

$$\begin{array}{ccc} & \vdots & \\ PCPCA & & CPCPCA \\ PCA & & CPCA \\ \mathfrak{A} & & CA \longleftarrow \text{coanalytic} \\ \rotatebox{90}{$\subsetneq$} & & \rotatebox{90}{$\subsetneq$} \\ & \mathfrak{B} & \end{array}$$

<u>Rephrased:</u> At what level of the projective hierarchy do sets stop to have the p.s.p.

<u>Theorem</u> In $\mathbb{L}$, there is a PCA set without the p.s.p.

In the remaining two lectures, we give all definitions and prove the theorem.

Lecture $\underline{\overline{XII}}$: Both on Zoom & in the Multimedia Room Geom 414