TENTH LECTURE of The Constructible Universe

17 June 2024

So far — We proved:

If $M \models ZF$, then there is $\mathbb{L} \subseteq M$ s.t. $\mathbb{L} \models ZFC + GCH$.

Method of Condensation:

$A \subseteq \mathbb{N}$ $\rightsquigarrow$ there is $\alpha < \omega_1$ s.t.

$A \in \mathbb{L}$ — $A \in \mathbb{L}_\alpha$ — MINIATURISATION

Today — Applications: $\diamondsuit$.

DIAMOND. — KARO

Ronald Jensen

Jensen giving a lecture in 2007

Born	April 1, 1936 (age 88)
Nationality	American
Alma mater	University of Bonn

$\diamondsuit$ is a strengthening of CH, holds in $\mathbb{L}$, & allows us to solve open problems

$\longrightarrow$ Suslin's problem

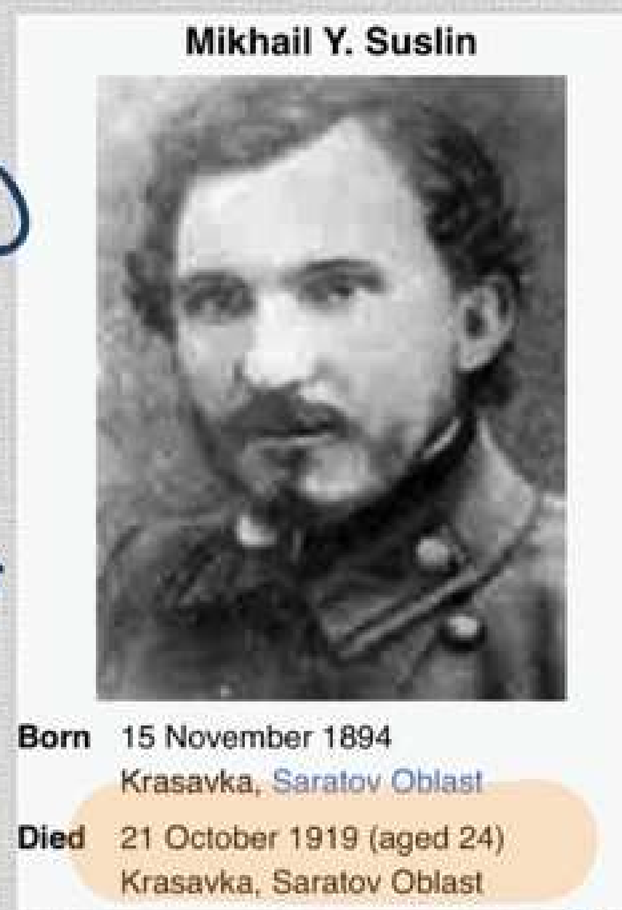
§ 20 Suslin's Problem

$(X, \leq)$ total order

DENSE $\forall x \forall y (x<y \to \exists z\ x<z<y)$

UNBOUNDED $\forall x \exists y \exists z\ y<x<z$

COMPLETE if every bounded subset Z has a supremum (least upper bound) and an infimum (greatest lower bound)



SEPARABLE if there is a countable dense subset.

[Found "Lebesgue's error": L. had claimed that the Borel sets are closed under cts images $\longrightarrow$ more on this later.]

Clearly $(\mathbb{R}, <)$ are dense, unbounded, complete, & separable [witnessed by $\mathbb{Q}$]

Fact $(X, \leq)$ is dense, unbounded, complete, & separable

$$\Longleftrightarrow$$

$$(X, \leq) \cong (\mathbb{R}, <).$$

Proof sketch Behind this is Cantor's theorem on $\mathbb{Q}$.

Theorem (Cantor)

Every ctble dense and unbounded total order is isomorphic to $\mathbb{Q}$.

[Proof maybe later]

Now take $(X,<)$ which is
- unbounded
- dense
- complete

& separable, i.e., $Q \subseteq X$ countable dense in X.

So Q is a countable dense unbounded order, thus by Cantor iso to $\mathbb{Q}$.

Let $f: \mathbb{Q} \longrightarrow Q$ be iso. with $q_n \in \mathbb{Q}$.

For $x \in \mathbb{R}$ pick $(q_n; n \in \mathbb{N})$ s.t. $q_n \longrightarrow x$ and $q_n < q_{n+1}$ for $n \in \mathbb{N}$ and let

$$\bar{f}(x) := \sup\{f(q_n); n \in \mathbb{N}\}$$

Show that this is well-defined & defines an isomorphism.

q.e.d. (Sketch)

$(X,<)$ has the countable chain condition (c.c.c.) if every I of nonempty pairwise disjoint intervals is countable.

Remark. It should be called "countable antichain condition", but ccc is such a nice acronym.

Proposition If X is separable, then X has the c.c.c.

Pf. Let I be a collection of pw. disjoint nonempty intervals; if $i \in I$ there are $x, y \in X$ s.t. $i = (x, y) = \{z ; x < z < y\}$.

By density of Q, the countable dense subset, each $i \in I$ contains an element of Q. Disjointness implies that they are pw. different.

So I get an injection from I to Q, so I is countable.

q.e.d.

SUSLIN asked

Is every dense, c.c.c., complete, unbounded total order iso to $(\mathbb{R}, <)$?

Def. A total order is called a Suslin line if it is dense, c.c.c., complete, unbounded and not iso to $(\mathbb{R}, <)$.

SH (Suslin's hypothesis) There are no Suslin lines.

[Souslin]

Answer SH cannot be solved in ZFC; there are models of ZFC+SH and models of ZFC+ $\underbrace{\neg SH}_{\text{there is a Suslin line}}$.

It turns out that $\boxed{\mathbb{L} \models \neg SH.}$

§ 21 Diamond.

Ronald Jensen proved $\mathbb{L} \models \neg SH$ and later isolated the combinatorial principle behind that proof, calling it $\diamond$ ("diamond").

If α is a limit ordinal, we call

$A \subseteq \alpha$ <u>unbounded</u> if $\forall \beta \in \alpha \, \exists \gamma \in A$ $\gamma \geq \beta$.

$A \subseteq \alpha$ <u>closed</u> if whenever $A \cap \gamma$ is unbounded in γ, then $\gamma \in A$.

$A \subseteq \alpha$ <u>club</u> closed & unbounded

If $\gamma < \alpha$, then $\{\beta \in \alpha ; \beta > \gamma\}$ is club.

All club sets are "big" subsets of α.

$A \subseteq \alpha$ <u>stationary</u> if for all C club in α $A \cap C \neq \emptyset$

Proposition Every stationary set is unbounded.

Proof. $A \subseteq \alpha$ stationary and $\gamma \in \alpha$, then take $C := \{\beta \,;\, \beta \geq \gamma\}$ which is club by remark. Then $A \cap C \neq \emptyset$, so A contains $\beta \geq \gamma$.

q.e.d.

Stationary sets are still somewhat big.

Definition Let $(A_\alpha ; \alpha < \omega_1)$ be a sequence of sets s.t. $A_\alpha \subseteq \alpha < \omega_1$.
Then $(A_\alpha ; \alpha < \omega_1)$ called a DIAMOND SEQUENCE if for all $A \subseteq \omega_1$, the set

$$A^* := \{\alpha < \omega_1 \,;\, A \cap \alpha = A_\alpha\}$$

is stationary.

A diamond seq. is a fixed seq. that predicts any set $A \subseteq \omega_1$ on the initial segments of a "somewhat big" set.

Jensen's Diamond Principle $\diamond$ states:
"There is a Diamond Sequence".

Theorem (Jensen) w/o proof.

$$\diamondsuit \implies \neg SH.$$

Proposition $\diamondsuit \implies CH$.

Proof. If $A \subseteq \mathbb{N} \subseteq \omega_1$. If $(A_\alpha ; \alpha < \omega_1)$ is a diamond sequence:

$$A^* = \{ \alpha < \omega_1 ; A \cap \alpha = A_\alpha \}$$

is stationary. (Thus unbounded by Prop.)

i.e. $A^* \cap C \neq \emptyset$ for all C club.

So there is $\alpha > \omega$ s.t. $\underset{\overset{\shortparallel}{A}}{A \cap \alpha} = A_\alpha$.

$$\mathcal{P}(\mathbb{N}) = \{ A_\alpha ; A_\alpha \subseteq \mathbb{N} \}$$

But that means $|\mathcal{P}(\mathbb{N})| \leq \aleph_1$.

q.e.d.

§22 $\diamondsuit$ in $\mathbb{L}$.

Theorem (Jensen 1972)

$$\mathbb{L} \models \diamondsuit.$$

Proof. Proof idea is miniaturisation: The proof of $2^{\aleph_1} = \aleph_2$ in $\mathbb{L}$ (§19) shows that all subsets of $\aleph_1$ have $\mathbb{L}$-rank $< \omega_2$. Every relevant lives in $\mathbb{L}_{\omega_2}$.
But "$A \cap \alpha = A_\alpha$" is something that is reflected in a ctble substructure. Use that to construct the A_α.

Define

$\Phi(\alpha, X, Y, s) :\Longleftrightarrow$ α is a limit ordinal, $X \subseteq \alpha$, $Y \subseteq \alpha$, Y is a club set, $\mathrm{dom}(s) = \alpha$ and $\forall \eta \in Y \;\; X \cap \eta \neq s(\eta)$.

Note that Φ is absolute for transitive models.

[Φ says that Y witnesses that X hasn't been guessed correctly so far.]

Y takes the role of C
X takes the role of A

"$A^* \cap C = \emptyset$".

We define functions S (and auxiliary fns A and C) by recursion on ω_1:

$S(\alpha) = (\emptyset, \emptyset)$ for any α successor or 0

$S(\alpha) = (X, Y) \rightsquigarrow A(\alpha) := X$
$C(\alpha) := Y$

If α is limit:
If there is no pair (X, Y) s.t. $\Phi(\alpha, X, Y, A\restriction\alpha)$ holds

then $S(\alpha) := (\emptyset, \emptyset)$

o/w let X, Y be the $\mathbb{L}$-least pair s.t. $\Phi(\alpha, X, Y, A\restriction\alpha)$ holds.

uses $V = L$

This is the global wellordering of $\mathbb{L}$ from the section on AC which is defined by an absolute fmla.

This definition is a recursion with absolute recursion steps, so it itself has an absolute formula.

$\Sigma(\alpha, X, Y)$ for $S(\alpha) = (X, Y)$

Thus Σ is absolute for transitive models.

CLAIM $(A(\alpha) ; \alpha < \omega_1)$ is a diamond sequence.

Towards a contradiction, let's assume it's not:

there is $E \subseteq \omega_1$ and $D \subseteq \omega_1$ club s.t. $D \cap E^* = \emptyset$

$= D \cap \{\alpha < \omega_1 ; D \cap \alpha = A(\alpha)\}$.

Note that all of our objects: D, E, A, C, S, ω_1 are all elements of $L_{\omega_2} = H_{\omega_2}$. In particular,

$H_{\omega_2} \models$ "there is (E,D) s.t. $\Phi(\omega_1, E, D, A)$"

Let's w.l.o.g. assume (E,D) is the $\mathbb{L}$-least such pair. Then

$H_{\omega_2} \models$ "(E,D) is the $\mathbb{L}$-least pair s.t. (*) $\Phi(\omega_1, E, D, A)$".

Note that A, C, S, ω_1 are all definable in H_{ω_2}; and so are (E,D) by (*)

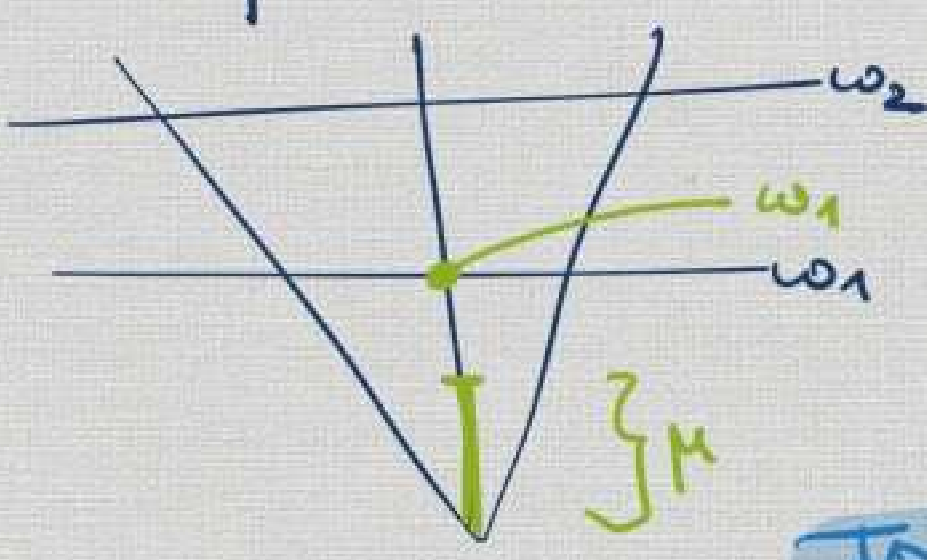
This means that if $M \prec H_{\omega_2}$, $E, D, A, C, S, \omega_1 \in M$.

Let $H \prec H_{\omega_2}$ countable. By , we have that $E, D, A, C, S, \omega_1 \in H$.

<u>Claim 1</u> $H \cap \omega_1$ is a countable ordinal.

[Clearly countable, since H is countable.
So only need to show transitivity:
if. $\xi < \eta$ s.t. $\xi \notin H$, $\eta \in H$.
Then $H_{\omega_2} \models$ there is a surj. from ω to η.
By elementarity,
$H \models$ there is a surj. from ω to η.
By Tarski-Vaught. there is $f \in H$ s.t.
$H_{\omega_2} \models f : \omega \twoheadrightarrow \eta$.
Find n s.t. $f(n) = \xi$.
$H_{\omega_2} \models \exists x (x = f(n))$,
so $H \models \exists x (x = f(n))$
with only witness ξ, so $\xi \in H$.]

Write $\mu := H \cap \omega_1$ with $\mu < \omega_1$.



This means that in the Mostowski collapse of H, with collapse function π, we have $\pi(\omega_1) = \mu$.

TO BE CONTINUED.