The Constructible Universe

LECTURE VII

27 May 2024

OVERALL GOAL

Given $M \models ZF$, construct $\boxed{N \subseteq M}$ s.t. $N \models ZFC + CH$.

INNER MODEL

Gödel's Idea The constructible universe, i.e., the minimal model of set theory that includes the absolutely necessary sets.

Template

von Neumann hierarchy

$$V_0 := \emptyset$$
$$V_{\alpha+1} := \boxed{\mathcal{P}(V_\alpha)}$$
$$V_\lambda := \bigcup_{\alpha<\lambda} V_\alpha$$

INSTEAD

$$L_0 := \emptyset$$
$$L_{\alpha+1} := \boxed{\mathcal{D}(L_\alpha)}$$
$$L_\lambda := \bigcup_{\alpha<\lambda} L_\alpha$$

where $\mathcal{D}(X)$ is the "definable powerset".

$\mathcal{D}(X) := \{ y \subseteq X \,;\,$ there is φ and $\vec{p} \in X^n$

$z \in y \iff X \models \varphi(z, \vec{p}) \}$

Necessary to do "definable over X" since definability is not definable.

The operation $\mathcal{D}$ is absolute for transitive models;

since absoluteness for transitive models is closed under recursive definitions, this means that

$\boxed{x \in L_\alpha}$ and $\boxed{\exists \alpha\, x \in L_\alpha}$

are absolute for transitive models.

Thus. if $\boxed{L = \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha}$

Then if M is a transitive set, then

$\boxed{L \cap M \subseteq M}$

Moreover, if $\alpha := M \cap \mathrm{Ord}$, $L \cap M = L_\alpha$.

So: If $L \models ZF$, then L is the minimal transitive model of ZF.

§12: $L \models$

$$\boxed{\begin{array}{l} \text{Ext} \\ \text{Found} \\ \text{Inf} \\ \text{Pair} \\ \text{Un} \\ \text{Pow} \end{array}}$$

- Ext ✓ generally true for transitive classes
- Found ✓ generally true
- Inf ✓ $\omega \in L_{\omega+1}$
- Pair ✓] easy
- Un ✓] easy
- Pow ✓ — slightly harder

So far, we do not know the rank of $\wp(x)$ in relation to the rank of x.

What about $\boxed{\text{Sep \& Rep}}$?

Lecture VI, page 10

Separation: This simple technique won't work & here's the reason:

Suppose $x \in L_\alpha$, φ formula, $\vec{p} \in L_\alpha^{<\omega}$

Want $\boxed{\{y \in x;\ L \models \varphi(y, \vec{p})\} = S}$ to be in L.

What we get is unfortunately only

$$\boxed{\{y \in x;\ L_\alpha \models \varphi(y, \vec{p})\} = S'}$$

If φ is not absolute, then S' might be very different from S and so the fact that $S' \in \mathcal{D}(L_\alpha)$ doesn't mean much.

... to ensure that φ

§13 Lévy Reflection Theorem

Lecture VI, page 11

Def. An assignment $\alpha \longmapsto Z_\alpha$ is called a **HIERARCHY** if

1. Z_α is a transitive set
2. $\mathrm{Ord} \cap Z_\alpha = \alpha$
3. $\alpha < \beta \Longrightarrow Z_\alpha \subseteq Z_\beta$
4. λ limit $\Longrightarrow Z_\lambda = \bigcup_{\alpha<\lambda} Z_\alpha$

If we have a hierarchy, can define a proper class $Z = \bigcup_{\alpha \in \mathrm{Ord}} Z_\alpha$ [by 2 $\mathrm{Ord} \subseteq Z$] and

$$\rho_Z(x) := \min\{\alpha ; x \in Z_\alpha\}$$

for any $x \in Z$.

Lévy Reflection Theorem

If $M \models ZF$, $\alpha \longmapsto Z_\alpha$ is a hierarchy, and φ is a formula, then there are unboundedly many ϑ s.t. φ is absolute between Z_ϑ and Z.

§14 ZF in the constructible universe

Goal of this section

Prove Sep and Rep in $\mathbb{L}$. .

Theorem $\mathbb{L} \models$ Sep.

Proof. Fix $x \in L_\alpha$, $\varphi \in \mathrm{Fml}$, $\vec{p} \in L_\alpha^{<\omega}$.

We need to show that

$$S := \{ y \in x ;\ \mathbb{L} \models \varphi(y, \vec{p}) \}$$

is in $\mathbb{L}$.

As before, for any $\gamma \geq \alpha$, we know that

$$S^\gamma := \{ y \in x ;\ \mathbb{L}_\gamma \models \varphi(y, \vec{p}) \} \in \underbrace{\mathcal{D}(L_\gamma)}_{= L_{\gamma+1}}$$

So $S^\gamma \in \mathbb{L}$.

In general, $S^\gamma \neq S$. Need to find γ s.t.

$S^\gamma = S$.

Remember that L is a hierarchy in the sense of the LRT:

So, there is $\vartheta \geq \alpha$ s.t. φ is absolute between L_ϑ and L. Thus: $S^\vartheta = S$.

q.e.d.

Remark As in the proof of power set, we do not know the size of δ with respect to the original α, i.e., we don't know the $\mathbb{L}$-rank of the separation instance.

Theorem $\mathbb{L} \models$ Repl.

Proof. Suppose φ is a formula that is functional in $\mathbb{L}$, i.e., for all $\vec{p}, x, y, y'$
$\mathbb{L} \models \varphi(x, y, \vec{p})$ & $\mathbb{L} \models \varphi(x, y', \vec{p})$
then $y = y'$.

Need to show: if $Z \in \mathbb{L}$, then fix α s.t. $Z \in \mathbb{L}_\alpha$.

$$R := \{y \,;\, \exists x \in Z \ \mathbb{L} \models \varphi(x, y, \vec{p})\} \in \mathbb{L}.$$

If $Z \in \mathbb{L}$, find α s.t. $Z \in \mathbb{L}_\alpha$.

Consider for $\gamma \geq \alpha$

$$R^\gamma := \{y \,;\, \exists x \in Z \ \mathbb{L}_\gamma \models \varphi(x, y, \vec{p})\}$$

Added problem Since φ is only functional in $\mathbb{L}$, and not necessarily in $\mathbb{L}_\gamma$, we do not even know that R^γ is even a set ...

Aside. [Example of a fmla that is functional in $\mathbb{L}$, but not for all $\mathbb{L}_\alpha$.]

$\varphi(x,y) :\Longleftrightarrow$ there is a large^{t} ordinal

OR

there is no largest ordinal

AND $x=y$.

In $\mathbb{L}$, φ describes the identity class function and therefore, the fmla is functional.

In $\mathbb{L}_{\gamma+1} \models$ there is a largest ordinal,

so $\mathbb{L}_{\gamma+1} \models \forall x \forall y\, \varphi(x,y)$.

Thus φ is not functional in $\mathbb{L}_{\gamma+1}$.

Want to have

$$R := \{y ;\ \exists x \in Z\ \mathbb{L} \models \varphi(x,y,\vec{p})\}$$

Use LRT to obtain $\vartheta \geq \alpha$ where $Z \in \mathbb{L}_\alpha$ s.t. φ is absolute between $\mathbb{L}_\vartheta$ and $\mathbb{L}$.

This implies: φ is functional for $\mathbb{L}_\vartheta$.

Thus by Replacement in M,

$$R^\vartheta := \{y ;\ \exists x \in Z\ \mathbb{L}_\vartheta \models \varphi(x,y,\vec{p})\}$$

is a set. By absoluteness $R^\vartheta = R$.

And $R^\vartheta \in \mathcal{D}(L_\vartheta) = L_{\vartheta+1}$. q.e.d.

<u>Corollary</u> $\mathbb{L} \models ZF$.

Therefore, $\mathbb{L}$ is the minimal model of ZF.

AXIOM OF CONSTRUCTIBILITY

$$\boxed{\forall x \exists \alpha \ x \in L_\alpha}$$

We write $V=L$ for this axiom.

Note that it looks like an equation, but it's just shorthand for $\forall x \exists \alpha \ x \in L_\alpha$.

<u>Corollary</u> $\mathbb{L} \models ZF + V=L$.

§ 15 The Axiom of Choice

We'll show: if $M \models ZF$ and $\mathbb{L}$ is built inside M, then $\mathbb{L} \models ZFC$.

We prove this by providing well-orderings of all of the L_α. More precisely, we'll provide a class fn

$$\prec : \alpha \longmapsto \prec_\alpha$$

s.t. $(L_\alpha, \prec_\alpha)$ is wellorder and if $\alpha < \beta$, then $\prec_\beta$ is an end-extension of $\prec_\alpha$ and $\prec_\alpha \in \mathbb{L}$.

Remark This is much stronger than just AC: we are producing a GLOBAL WELL-ORDER of $\mathbb{L}$.

Corollary Suppose we have produced such a class function $\prec$. Then $\mathbb{L} \models AC$.

Proof. Take any $x \in \mathbb{L}$. So there is α s.t. $x \in \mathbb{L}_\alpha$; so $x \subseteq \mathbb{L}_\alpha$. Thus $(\mathbb{L}_\alpha, \prec_\alpha)$ is wellordered and so is

$$(x, \prec_\alpha \cap (x \times x)).$$

q.e.d.

Theorem Such a class function $\prec$ exists.

Proof. By induction on α.

① If $\alpha = 0$, $L_\alpha = L_0 = \emptyset$, so nothing to show.

② If α is a limit ordinal. By IH have $\prec_\beta$ s.t. $(L_\beta, \prec_\beta)$ is a wellorder for each $\beta < \alpha$ and $\beta < \beta'$, then $\prec_{\beta'}$ agrees with $\prec_\beta$ on L_β.

Define $x \prec_\alpha y :\Longleftrightarrow \exists \beta < \alpha$ s.t. $x \prec_\beta y$.

③ Suppose $\alpha = \gamma + 1$. By IH have $\prec_\gamma$ s.t. $(L_\gamma, \prec_\gamma)$ is a wellorder.

What is $L_{\gamma+1}$?

$$\overset{\shortparallel}{\mathcal{D}}(L_\gamma)$$

Elements of L_γ

Non-elements of L_γ.

Clear: (a) on L_γ, order must be $\prec_\gamma$

(b) all elts of L_γ come before all elements of $L_{\gamma+1} \setminus L_\gamma$.

These are given by φ & $\vec{p}$

$x \in \mathcal{D}(L_\gamma) \Longleftrightarrow x = \{ y \in L_\gamma ; L_\gamma \models \varphi(y, \vec{p}) \}$

For each $x \in \mathcal{D}(L_\gamma)$ there are φ $\vec{p}$

s.t. $x = \{y \in L_\gamma ; L_\gamma \models \varphi(y, \vec{p})\}$

Fml $L_\gamma^{<\omega}$

We have that Fml is wellordered in order type ω;

and the wellordering $<_\gamma$ lifts by "short-lex" $\leq_{<\omega}$ to a wellordering $<_{lex}$ of L_γ.

First by length, then lexico-graphically.

Thus define a wellorder $<^*$ on $\mathrm{Fml} \times L_\gamma^{<\omega}$, again lexicographically

If $x \in L_{\gamma+1} \setminus L_\gamma$, we have that

$$W := \{(\varphi, \vec{p}) ; x = \{y \in L_\gamma ; L_\gamma \models \varphi(y, \vec{p})\}\}$$

$$\neq \emptyset,$$

so we can define

$(\varphi_x, \vec{p}_x)$ to be the $<^*$-minimal element of W.

Now define

$$x <_{\gamma+1} y \;:\Longleftrightarrow\; \begin{array}{c} x,y \in L_\gamma \wedge x <_\gamma y \\ \text{OR} \\ x \in L_\gamma \wedge y \notin L_\gamma \\ \text{OR} \\ x,y \notin L_\gamma \wedge \\ (\varphi_x, \vec{p}_x) <^* (\varphi_y, \vec{p}_y) \end{array}$$

Clearly, $<_{\gamma+1}$ is a wellorder of $L_{\gamma+1}$ and satisfies the requirements of the induction hypothesis by construction.

<u>Remark</u> $<_\alpha$ is defined by recursion and therefore definable over a sufficiently large L_ϑ s.t. $<_\alpha \in \mathcal{D}(L_\vartheta) = L_{\vartheta+1}$.

q.e.d.

Theorem For all $\alpha \geq \omega$, $|L_\alpha| = |\alpha|$.

Proof. By induction.

Clearly, $|L_\omega| = |V_\omega| = \aleph_0 = |\omega|$.

Successor case Suppose $\alpha = \gamma + 1$.

IH: $|L_\alpha| = |\alpha|$.

$$\begin{aligned} |\alpha| = |L_\alpha| \leq |L_{\alpha+1}| &= |\mathcal{D}(L_\alpha)| \\ &\leq |\mathrm{Fml} \times L_\alpha^{<\omega}| \\ &= |\mathrm{Fml}|\,|L_\alpha^{<\omega}| \\ &= \aleph_0 \cdot |L_\alpha| = \aleph_0 \cdot |\alpha| \\ &= |\alpha| \\ &= |\alpha+1|. \end{aligned}$$

Limit case Suppose α limit.

[IH: $|L_\beta| = |\beta| \leq |\alpha|$]

$$\alpha \subseteq L_\alpha = \bigcup_{\beta<\alpha} L_\beta$$

$$|\alpha| \leq |L_\alpha| \leq |\alpha|\,|\alpha| = |\alpha|.$$

Remark. The cardinal arithmetic in this proof used the Axiom of Choice!!!

q.e.d.

Remark We had already seen that

$$\left.\begin{array}{l} L_{\omega+1} \text{ is countable} \\ V_{\omega+1} \text{ is uncountable} \end{array}\right\}$$

thus $L_{\omega+1} \neq V_{\omega+1}$.

But now we see that L_α (for $\alpha < \omega_1$) is countable

$$L_\alpha \neq V_\alpha \quad \text{if } \omega < \alpha < \omega_1$$

but also $V_{\omega+1} \not\subseteq L_\alpha$.

This matters since in $\mathbb{Z}$, $V_{\omega+1}$ is the size of $\mathbb{R}$ and identifying where it lives in the constructible hierarchy is crucial for determining the size of $2^{\aleph_0}$.

§16 The condensation sentence

This is a crucial preparation for the proof of CH: the main step in that proof will be called ~~the~~ condensation lemma.

If we write $\Phi(x,\alpha)$ for $x \in L_\alpha$, we proved that Φ is absolute for transitive models of ZF.

We proved more:

For Δ_0 fmlas, we proved that they are absolute for trs models.

If $\boxed{ZF \vdash \varphi \longleftrightarrow \psi}$ and φ is absolute for trs models, then ψ is absolute for trs models of ZF.

By compactness, there is a finite $T_\psi \subseteq ZF$ s.t.

$T_\psi \vdash \varphi \longleftrightarrow \psi$.

Thus ψ is absolute for trs models of T_ψ.

Also "ordinal" does not need all of ZF to be absolute, but only Foundation.

If F_0, F_S, F_L are absolute, then

$$H(0,x) := F_0(x)$$
$$H(\alpha+1,x) := F_S(H(\alpha,x),x)$$
$$H(\lambda,x) := F_L(H\restriction\lambda\times\{x\},x).$$

If T_0, T_S, T_L are finite subsets of ZF s.t. F_0, F_S, F_L are absolute for trs models of T_0, T_S, T_L

and $T \subseteq ZF$ finite s.t. T proves the preservation of abs. under recursion, then

$T_0 \cup T_S \cup T_L \cup T =: T_H$ has the property that H is absolute for transitive models of T_H.

<u>We obtain</u>. Each of the notions φ we proved to be absolute for transitive models of ZF is actually absolute for transitive models of $T_\varphi \subseteq ZF$ finite for a set T_φ that we could determine if we wanted to.

If $\Phi(x,\alpha)$ is the formula describing $x \in L_\alpha$, there is some finite T_Φ s.t. Φ is absolute for trs models of T_Φ.

Definition

the condensation sentence

(NOT QUITE: SEE BELOW)

$$\sigma := \bigwedge T_\Phi \wedge \text{Foundation} \wedge \underline{V=L}$$

$\forall x \exists \alpha \Phi(x,\alpha)$

The following is a correction added after Lecture VIII:

The Recursion Theorem proves that $\alpha \longmapsto \mathbb{L}_\alpha$ is a function class, i.e., that for each α, there is a set $\mathbb{L}_\alpha$ and there is a formula Φ^* s.t. $x = \mathbb{L}_\alpha \iff \Phi^*(x,\alpha)$.

This means that $ZF \vdash \forall \alpha \exists x \Phi^*(x,\alpha)$.

By compactness, a finite subset of ZF, say T_{Φ^*}, is sufficient to prove

$$T_{\Phi^*} \vdash \forall \alpha \exists x \Phi^*(x,\alpha)$$

Finally, $ZF \vdash$ "there is no largest ordinal"

$$\forall \alpha \exists \beta\ \beta > \alpha$$

and, once more, there is a finite subset of ZF, say T_{ord} s.t. $T_{ord} \vdash \forall \alpha \exists \beta\ \beta > \alpha$.

Let

$$\sigma^* := \bigwedge T_{\Phi} \wedge \bigwedge T_{\Phi^*} \wedge \bigwedge T_{\mathrm{ord}} \wedge \mathrm{Found} \wedge \mathbb{V} = \mathbb{L}$$

We now call σ^* the **CONDENSATION SENTENCE**

$$\boxed{\text{\underline{Theorem} If } X \text{ is transitive, } X \models \sigma^*, \text{ and } \alpha := X \cap \mathrm{Ord}, \text{ then } X = \mathbb{L}_\alpha.}$$

The proof will be given in Lecture $\underline{\mathrm{VIII}}$.