VIII Eighth Lecture

THE CONSTRUCTIBLE UNIVERSE

3 June 2024

Lecture VII — We proved that if $M \models ZF$ and L is built inside M, then $L \models ZFC$.

Corollary. $Con(ZF) \Longrightarrow Con(ZFC)$.

In other words: it is impossible to refute AC from ZF (unless ZF is inconsistent).

$\Phi(\alpha, x) :\Longleftrightarrow x \in \mathbb{L}_\alpha$

Also:

Note that this was corrected after the lecture: the incorrect claim here was that σ (not σ^*) is enough. This was fixed in the lectures and is now corrected here.

Finite part of ZF enough to show that Φ is absolute

Finite part of ZF enough to show that L_α exists

Finite part of ZF implying no largest ordinal

Let

$$\sigma^* := \bigwedge T_\Phi \wedge \bigwedge T_{\Phi^*} \wedge \bigwedge T_{ord} \wedge \text{Found} \wedge \mathbb{V} = \mathbb{L}$$

We now call σ^* the CONDENSATION SENTENCE

$$\boxed{\textbf{Theorem} \text{ If } X \text{ is transitive, } X \models \sigma^*, \text{ and } \alpha := X \cap Ord, \text{ then } X = \mathbb{L}_\alpha.}$$

The proof will be given in Lecture VIII.

Proof of the THEOREM

$X \models \sigma$ $\quad\quad$ $\sigma = \bigwedge T_{\Phi} + \text{Found} + \boxed{V=L}$

$\Updownarrow$

$\forall x \exists \gamma\, \Phi(\gamma, x)$

X transitive

$\alpha := \mathrm{Ord} \cap X$

(1) Since $X \models T_{\Phi}$ $\Longrightarrow$ Φ is absolute between X and M.

(2) Since $X \models \text{Found}$ $\Longrightarrow$ "is an ordinal" is absolute between X & M

Need to show $X = L_{\alpha}$.

Comment: So σ is sufficient to get $X \subseteq L_{\alpha}$.

First prove $X \subseteq L_{\alpha}$.

Let $x \in X$; since $X \models \sigma$ $\Longrightarrow$ there is an ordinal γ s.t. $\Phi(\gamma, x)$.

$X \models \Phi(\gamma, x)$

$\Longrightarrow$

$M \models \Phi(\gamma, x)$

$\Longrightarrow x \in L_{\gamma}$.

$\subseteq L_{\alpha}$.

[Note that $M \models \gamma$ is an ordinal $\Longleftrightarrow X \models \gamma$ is an ordinal AND $M \models \Phi(\gamma, x) \Longleftrightarrow X \models \Phi(\gamma, x)$.]

But $\gamma \in \mathrm{Ord} \cap M = \alpha$

$\Longrightarrow \gamma < \alpha$.

Thus $X \subseteq L_{\alpha}$.

Now prove $L_\alpha \subseteq X$. $(*)$

For this, we need more for our condensation sentence σ:

This was the moment in the lecture when we switched from σ to σ^*.

Consider the proof (an instance of the Recursion Theorem) that for all α there is some L_α. This proof needs a finite number of axioms of ZF, call that set Φ.

$$\sigma^* := \bigwedge \Phi \wedge \sigma \wedge \text{"there is no largest ordinal"}.$$

$$X \models \sigma^* \implies X \models \forall \gamma \exists z \; z = L_\gamma.$$

In order to prove $(*)$, it's enough to show $L_\gamma \in X$ for all $\gamma < \alpha$.

That's just what the previous line says.

q.e.d.

This little box was triggered by the question why it is surprising that being L_α can be characterised.

It's clear that "$X = L_\alpha$" can be expressed by a formula, but not clear ~~that~~ it can be AXIOMATISED.

The theorem shows that $X = L_\alpha$ is finitely axiomatisable on tos sets.

Definability

Being L_α is definable:
$\iff$ there is φ s.t.
$X = L_\alpha \iff M \models \varphi(X)$.

Being L_α is characterisable
$\iff$ there is σ s.t.
$X = L_\alpha \iff X \models \sigma$

"finitely axiomatisable".

More recap

von Neumann
$V_0 := \emptyset$
$V_{\alpha+1} := \mathcal{P}(V_\alpha)$
$V_\lambda := \bigcup_{\alpha<\lambda} V_\alpha$

In general, $L_\alpha \subsetneqq V_\alpha$. In particular, $L_\alpha \subsetneqq V_\alpha$ for all $\alpha<\omega_1$.

Constructible
$L_0 := \emptyset$
$L_{\alpha+1} := \mathcal{D}(L_\alpha)$
$L_\lambda := \bigcup_{\alpha<\lambda} L_\alpha$

Proved: $\rightarrow$ $|L_\alpha| = |\alpha|$ for all $\alpha \geq \omega$

In the proof on inaccessible cardinals, we proved: if κ is inaccessible, then f.a. $\alpha<\kappa$ $|V_\alpha| < \kappa$. (*)

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§ 17 Constructible ranks & inaccessibility

κ is inaccessible $\iff$ κ is regular & $\forall\lambda<\kappa\ 2^\lambda<\kappa$.

$\implies V_\kappa \models ZF$. (used (*))

Another hierarchy: Let κ be regular.

$$H_\kappa := \{ A \,;\, |tcl(A)| < \kappa \}$$

SETS OF HEREDITARY SIZE $< \kappa$.

For $\kappa \geq \omega$, almost all axioms of ZF hold in H_κ.

In general, $H_\kappa \models$ Replacement, in fact SECOND-ORDER REPLACEMENT:

If $F : H_\kappa \longrightarrow H_\kappa$ (*) and $x \in H_\kappa$

$|x| \leq |\mathrm{tcl}(x)| < \kappa$,

$|\{F(y) ; y \in x\}| \leq |x| < \kappa$.

$\{F(y) ; y \in x\} \subseteq H_\kappa$ (*)

Thus $\{F(y) ; y \in x\} \in H_\kappa$.

However, if κ is a successor cardinal, say $\kappa = \lambda^+$, then $\wp(\lambda) \notin H_\kappa$

[since $|\wp(\lambda)| = 2^\lambda \geq \lambda^+ = \kappa$.]

and $\wp(\lambda) \subseteq H_\kappa$,

thus $H_\kappa \models \neg$ Powerset.

Theorem $H_\kappa = V_\kappa \iff \kappa$ is inaccessible.

[Notice: the H_κ's are all models of Replacement; the V_κ's are all models of Powerset].

Proof. Show $H_\kappa \subseteq V_\kappa$.

Suppose $x \in H_\kappa$, then $|x| < \kappa$.

Observe that if x is transitive, then so is $\{\varrho(y) ; y \in x\}$.

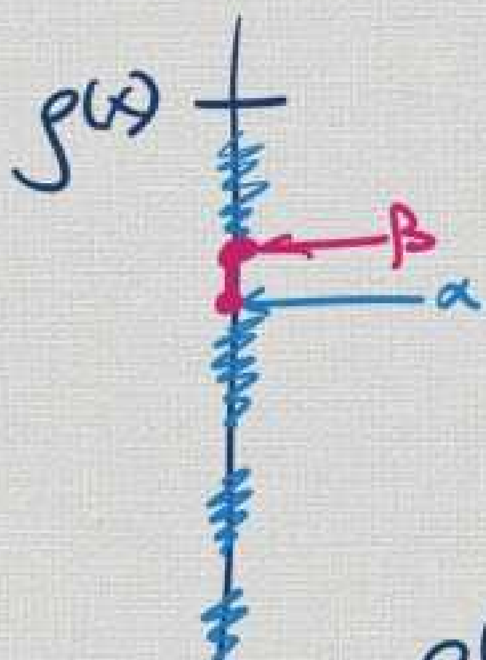
Let's prove this:

If x is transitive, then so is

$$\{\rho(y);\ y \in x\}.$$

Suppose not, so there is some $\alpha \in \rho(x)$ s.t. there is no $y \in x$ with $\rho(y) = \alpha$.

We know that



$$\rho(x) = \sup\{\rho(y)+1;\ y \in x\}.$$

So we find β least s.t. there is some $y \in x$ $\rho(y) = \beta$ and $\beta > \alpha$.

$$\rho(y) = \sup\{\rho(z)+1;\ z \in y\}$$
$$\leq \alpha$$

Contradiction!

So if $x \in H_\kappa$, $\{\rho(y);\ y \in tcl(x)\} \subseteq \kappa$ and is transitive

So it's an ordinal, say η.

By being in H_κ, $\eta < \kappa$.

But now $\rho(x) \leq \eta + 1 < \kappa$.

That proves $H_\kappa \subseteq V_\kappa$ for regular κ.

<u>Remark</u> It's not obvious from the definition of H_κ that it is a set. This follows from $H_\kappa \subseteq V_\kappa$ + separation.

Now show that κ inaccessible $\iff V_\kappa \subseteq H_\kappa$.

"$\Longrightarrow$". κ inacc. $\Longrightarrow$ if $x \in V_\kappa$,

so $x \in V_\alpha$ some $\alpha < \kappa$

then $x \subseteq V_\alpha$, so

$$|tcl(x)| \leq |V_\alpha| < \kappa$$

[by our lemma about inacc. cardinals; ⊛ on page 4].

"$\Longleftarrow$" Suppose $V_\kappa \subseteq H_\kappa$.

Only need to show that κ is a strong limit. Take $\lambda < \kappa$, then $V_{\lambda+1} \in V_\kappa \subseteq H_\kappa$,

so $|tcl(V_{\lambda+1})| = |V_{\lambda+1}| < \kappa$

$2^\lambda \leq$

q.e.d.

Theorem $V = L \implies L_\kappa = H_\kappa$ for all regular cardinals κ

Proof. $H_\omega = V_\omega = L_\omega$. ✓

Suppose $x \in L_\kappa$, there is $\alpha < \kappa$ $x \in L_\alpha$
$\implies tcl(x) \subseteq L_\alpha$

So $|tcl(x)| \leq |L_\alpha| \leq |\alpha| < \kappa$.

This shows $L_\kappa \subseteq H_\kappa$. (*)

Suppose $H_\kappa \not\subseteq L_\kappa$.

So find $x \in H_\kappa \setminus L_\kappa$ $\in$-minimal by Foundation.

Since $x \in H_\kappa$, $x \subseteq H_\kappa$.

So $x \subseteq H_\kappa \cap L_\kappa = L_\kappa$ (by (*))

because of $\in$-minimality of x

But $|x| < \kappa$, so for each $y \in x$ $\rho_L(y) < \kappa$,

and $\sup\{\rho_L(y) ; y \in x\} =: \eta < \kappa$.

And then $x \in L_{\eta+1}$.

q.e.d.

Corollary If κ is inaccessible and $V=L$, then $L_\kappa = V_\kappa = H_\kappa$ and

$$L_\kappa \models ZFC + V=L.$$

Theorem If κ is inaccessible, then there is a $\alpha < \omega_1$ s.t.

$$L_\alpha \models ZFC.$$

Proof. We use the technique to build ctble elementary (transitive) submodels:

Start with M and $A \subseteq M$; fix

Skolem function [choice fn that picks witnesses for true existential formulas in M]

Build

$H_0(A) := A$

$H_{n+1}(A) := H_n(A) \cup$ all witnesses of true (in M) existential fmlas with param. in $H_n(A)$

$\mathcal{H}(A) := \bigcup_{n\in\mathbb{N}} H_n(A)$

Tarski-Vaught: $\mathcal{H}(A) \preccurlyeq M$.

$|\mathcal{H}(A)| = \max(\aleph_0, |A|)$.

If $M \models$ Foundation & Extensionality, then we can collapse $\mathcal{H}(A)$ to a transitive set $X \subseteq M$.

We have an elementary embedding

$$\pi : (X, \in) \longrightarrow (M, \in)$$

and therefore $X \equiv M$.

If now κ is inaccessible and therefore $L_\kappa \models$ ZFC, then $L_\kappa \models \sigma^*$ [our condensation sentence]

So if we form X as above in L_κ, we get a countable $\underset{\text{transitive}}{X} \subseteq L_\kappa$ s.t.

$$X \equiv L_\kappa,$$

in particular $X \models \sigma^*$.

So by our Theorem, $X = L_\alpha$ for some α.

Now since $\underset{= L_\alpha}{X}$ is countable and $|L_\alpha| = |\alpha|$,

we know that $\alpha < \omega_1$.

q.e.d.

<u>Remark</u> In the "V-hierarchy" version of this proof you have to define $H^+_{\mathrm{un}}(A) := V_\beta$ where β is least s.t. all elts of $H_{\mathrm{un}}(A)$ are in V_β. Then $\mathcal{H}^+(A) = V_\gamma$ for some γ, but we lose control over cardinality. In general γ is a cardinal and quite big.

Note that this construction tells us that there are no meaningful implications between

$$\mathrm{IC} \Longleftrightarrow \exists\kappa\ \kappa \text{ is inaccessible}$$

& $V=L$.

To see that $V=L \not\Rightarrow \mathrm{IC}$:

Suppose there is an inacc. cardinal; let κ be the least such.

Then $V_\kappa \models ZFC + V=L$.

$$V_\kappa = L_\kappa$$

But if $\lambda < \kappa$ s.t. $V_\kappa \models \lambda$ is inaccessible, then $V \models \lambda$ is inaccessible [check that being inaccessible is absolute for models of the form V_α].

To see that $V=L \not\Rightarrow \neg\mathrm{IC}$.

Suppose there are inacc. cardinals. Being inaccessible is downwards absolute for transitive models, so force L inside $M \models ZFC+IC$ and get $L \models ZFC + V=L + IC$.

Next time: CH!