The Constructible Universe VI

13 May 2024

§11 Definition of the constructible universe

RECAP

"definable power set" is a slight misnomer

$\mathcal{D}(X) \subseteq \wp(X)$

Definition If X is a set, then let

$$\mathcal{D}(X) := \{A \subseteq X;\ A \text{ is definable with parameters over } X\} = \{A \subseteq X;\ N \models \Delta(A, X)\}$$

Sometimes called "the definable power set operation".

universe.

Reminder

$$V_0 := \emptyset$$
$$V_{\alpha+1} := \wp(V_\alpha)$$
$$V_\lambda := \bigcup_{\alpha<\lambda} V_\alpha$$

von Neumann hierarchy

[Forming this within N produces N since N ⊨ Foundation.]

The constructible hierarchy

History of letter L unknown.

$$L_0 := \emptyset$$
$$L_{\alpha+1} := \mathcal{D}(L_\alpha)$$
$$L_\lambda := \bigcup_{\alpha<\lambda} L_\alpha$$

Note that the $\mathcal{D}$-operator was shown to be absolute in §10.

Main Theorem of §10 said that transfinite recursion preserves absoluteness.

From lecture V

Lecture V

<u>Corollary</u> The statement

$$x \in L_\alpha$$

is absolute for transitive models of set theory.

<u>Definition</u> x is called <u>constructible</u> if there is $\alpha \in \mathrm{Ord}$ s.t. $x \in L_\alpha$.

<u>Corollary</u> If $M \subseteq N$ and $\mathrm{Ord} \cap M = \mathrm{Ord} \cap N$, then "$x$ is constructible" is absolute between M & N.

Thus if I write $L := \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha$, then L (the constructible sets) is contained in every model of ZF.

<u>Thus</u>. If we can show that $L \models ZF$, then L is the MINIMAL MODEL of ZF.

Main goal for today: $\underline{L \models ZF}$

Then: it is the minimal transitive model.

Consequences of our definition

Lecture V

Observe ① For all X, $\mathcal{D}(X) \subseteq \mathcal{P}(X)$.

② If X is transitive, then so is $\mathcal{D}(X)$.

[If $a \in \underbrace{b \in \mathcal{D}(X)}$. But every element of X is definable over X with parameters.

$\Rightarrow b \subseteq X$

$\Rightarrow a \in X$

$\Longrightarrow a \in \mathcal{D}(X)$.]

$\Longrightarrow$ By induction, we can prove

$\boxed{L_\alpha \subseteq V_\alpha}$

$\Longrightarrow$ By induction, each L_α is transitive.

Therefore, the class of constructible sets is a transitive class.

From the von Neumann hierarchy $\boxed{V_\alpha \cap \mathrm{Ord} = \alpha.}$

$L_\alpha \cap \mathrm{Ord} \subseteq \alpha$.

I claim $L_\alpha \cap \mathrm{Ord} = \alpha$.

Proof (by induction) True for $\alpha = 0$.

The limit case is trivial. Suppose $L_\alpha \cap \mathrm{Ord} = \alpha$ and show that $\alpha \in L_{\alpha+1}$.

[Then $\alpha \in L_{\alpha+1} \cap \mathrm{Ord} \subseteq V_{\alpha+1} \cap \mathrm{Ord} = \alpha+1$ which proves the claim.]

Note that α is the set of ordinals in L_α by IH, so

$$\{x \in L_\alpha ; L_\alpha \models x \text{ is an ordinal}\} = \alpha,$$

so $\alpha \in \mathcal{D}(L_\alpha) = L_{\alpha+1}$.

q.e.d.

<u>Corollary</u> The constructible sets form a proper class, containing all ordinals.

<u>Also</u> For all α, $L_\alpha \in L_{\alpha+1}$.

Because

$$L_\alpha = \{x \in L_\alpha ; L_\alpha \models x = x\},$$

so $L_\alpha \in \mathcal{D}(L_\alpha) = L_{\alpha+1}$.

Consider the relationship between the L- and the V-hierarchy:

$$V_0 = L_0 = \emptyset.$$

$V_n = L_n$ for $n \in \mathbb{N}$

[since finite sets can always be defined with parameters by listing their elements]

$$L_\omega = \bigcup_{n \in \omega} L_n = \bigcup_{n \in \omega} V_n = V_\omega.$$

What happens at $\alpha = \omega + 1$.

$V_{\omega+1} = \mathcal{P}(V_\omega)$ is uncountable; cardinality $2^{\aleph_0}$.

$L_{\omega+1} = \mathcal{D}(L_\omega)$ is <u>countable</u>

There is a surjection from $\mathrm{Fml} \times L_\omega^{<\omega}$ onto $\mathcal{D}(L_\omega)$:

$$(\varphi, \vec{p}) \longmapsto \{x \in L_\omega ;\ L_\omega \models \varphi(x, \vec{p})\}$$

Thus: $V_{\omega+1} \neq L_{\omega+1}$.

Now that we know that $|L_{\omega+1}| = \aleph_0$, we get surjection from $\mathrm{Fml} \times L_{\omega+1}^{<\omega}$ onto $\mathcal{D}(L_{\omega+1}) = L_{\omega+2}$.

Thus for all countable $\alpha < \omega_1$, $|L_\alpha| = \aleph_0$ by induction.

Since $\omega_1 \subseteq L_{\omega_1}$, this is <u>not</u> true anymore for L_{ω_1}.

<u>Remark</u> $L_{\omega_1} = \bigcup_{\alpha < \omega_1} L_\alpha$, so $|L_{\omega_1}| = \aleph_1$.

<u>Consequence</u> The hierarchies are different in the sense that there are α s.t. $L_\alpha \neq V_\alpha$, but that <u>not</u> mean that there are non-constructible sets.

Def. If x is constructible, let

$$\rho_L(x) := \min\{\alpha;\ x \in L_{\alpha+1}\}$$

be the constructible rank (the L-rank) of x.

If all sets were constructible, lots of sets in $V_{\omega+1}$ must have higher L-rank than Mirimanoff rank.

Remark The fact that we understand the size of L_α much better than the size of V_α; in particular, we know at least one L-rank, viz., L_{ω_1}, with cardinality $\aleph_1$ will be relevant later when we get to CH.

§ 12 Simple axioms of ZF in the constructible universe

$\boxed{ZF}$

- Ext
- Found

true in all transitive models, so we get them in the constructible universe

- Pair
- Un

Inf — true since $\omega \in L_{\omega+1}$ and the fmla defining it is absolute

- Pow
- Sep
- Repl

WORKING IN SOME $M \models ZF$, BUILDING $L = \bigcup L_\alpha \subseteq M$ defbl TRANSITIVE SUBCLASS.

AC [left out & discussed in Lecture VII.]

(a) Pairing

Suppose $x, y \in L$, $x \in L_\alpha$, $y \in L_\beta$
wlog $\alpha \geq \beta$, so $x, y \in L_\alpha$.

Clearly, since $M \supseteq L$ is a model of ZF, $\{x, y\}$ exists and $\{x, y\} \subseteq L_\alpha$.

The formula $\varphi(z) := (z = x \vee z = y)$ defines $\{x, y\}$ over L_α, so

$$\{x, y\} \in \mathcal{D}(L_\alpha) = L_{\alpha+1}.$$

(b) Union.

Fix $\underline{x \in L_\alpha}$*. We have $\bigcup x$ is a set in M by ZF in M.

$$z \in \bigcup x \iff \overbrace{\exists y\,(\underline{y \in x}_{*} \wedge \underline{z \in y}_{*})}^{\varphi(z,x)}$$

* implies (since L_α is transitive) $\bigcup x \subseteq L_\alpha$.

φ defines $\bigcup x$ over L_α, by absoluteness for transitive models, absolutely,

so $\bigcup x \in \mathcal{D}(L_\alpha) = L_{\alpha+1}$.

(c) Power set

Fix $x \in L_\alpha$. By ZF in M, there is $\mathcal{P}(x) \in M$.
The canonical candidate for the L-power set of x would be

$$P := \mathcal{P}(x) \cap L.$$

If $P \in L$, then absoluteness of the fmla "$y \subseteq x$" gives us that P is the L-power set of x.

If P is very large compared to α, then $L_{\alpha+1}$ may not be large enough to contain all of P.

For each $p \in P$, consider $\rho_L(p)$ and

$$\{\rho_L(p) ; p \in \underbrace{p(x) \cap L}_{\in M}\} =: R$$

This is a set in M by Replacement in M.

So, find γ s.t. $R \subseteq \gamma$

i.e. $p(x) \cap L \subseteq L_\gamma$.

This is defined by the absolute fmla $y \subseteq x$.

$$\{y \in L_\gamma ; L_\gamma \models y \subseteq x\} \in \mathcal{D}(L_\gamma) = L_{\gamma+1}.$$

Remark We have NO IDEA (at the moment) what γ is. It could be $\alpha+1$ or α^+ or α^{++} or ... It depends on the relationship between L-rank & Mirimanoff rank (which we don't understand properly yet).

Seperation This simple technique won't work & here's the reason:

Suppose $x \in L_\alpha$, φ formula, $\vec{p} \in L_\alpha^{<\omega}$

Want $\{y \in x;\ L \models \varphi(y, \vec{p})\} = S$ to be in L.

What we get is unfortunately only
$$\{y \in x;\ L_\alpha \models \varphi(y, \vec{p})\} = S'$$

If φ is not absolute, then S' might be very different from S and so the fact that $S' \in \mathcal{D}(L_\alpha)$ doesn't mean much.

Hope There is no reason to assume that φ is absolute between L_α and L, but maybe we can find $\gamma > \alpha$ s.t. φ is absolute between L_γ and L.

$\longrightarrow$ Lévy Reflection Theorem

§13 The Lévy Reflection Theorem

Def. An assignment $\alpha \longmapsto Z_\alpha$ is called a HIERARCHY if

1. Z_α is a transitive set
2. $\mathrm{Ord} \cap Z_\alpha = \alpha$
3. $\alpha < \beta \implies Z_\alpha \subseteq Z_\beta$
4. λ limit $\implies Z_\lambda = \bigcup_{\alpha<\lambda} Z_\alpha$

If we have a hierarchy, can define a proper class $Z = \bigcup_{\alpha \in \mathrm{Ord}} Z_\alpha$ [by ② $\mathrm{Ord} \subseteq Z$] and

$$\rho_Z(x) := \min\{\alpha;\ x \in Z_\alpha\}$$

for any $x \in Z$.

Lévy Reflection Theorem

If $M \models ZF$, $\alpha \longmapsto Z_\alpha$ is a hierarchy, and φ is a formula, then there are unboundedly many ϑ s.t. φ is absolute between Z_ϑ and Z.

Remark We make no assumption that $Z \models ZF$ or even a fragment thereof. In fact $Z_\alpha := \alpha$ is a hierarchy in our sense.

The proof of the LRT is essentially an elaboration of

(a) Tarski-Vaught Lemma

(b) the use of TVL in constructing elementary submodels.

Lemma (Improved Tarski-Vaught Lemma)

Let $M \subseteq N$ be any models and Φ be a collection of fmlas closed under subfmlas. Then TFAE

(a) all fmlas in Φ are absolute between M & N

(b) for any $\varphi \in \Phi$ s.t. $\varphi = \exists x \psi$, we have

$$\forall \vec{y} \in M \ (\exists a \in N \ N \models \psi(a, \vec{y}) \Rightarrow \exists a \in M \ N \models \psi(a, \vec{y})).$$

Tarski-Vaught condition

Proof. Note that the original TVL is just the special case of $\Phi = \mathrm{Fml}$.
The proof of TVL proves this improved version; the closure of Φ under subformulas is needed to apply the IH to ψ which is a subformula of φ.

q.e.d.

Proof of LRT

Fix φ and let Φ be the closure of $\{\varphi\}$ under subformulas. Note that Φ is a finite set of formulas.

By the improved TVL, we only need to find ϑ s.t. Z_ϑ contains witnesses (in Z) to all existential formulas in Φ.

Fix some α — need to find $\vartheta > \alpha$ s.t. φ absolute between Z_ϑ and Z.

If $\exists x \psi \in \Phi$, check whether

$$Z \models \exists x \psi(x, \vec{p}) \quad \text{for some } \vec{p} \in Z_\alpha^{<\omega}$$

If not, set $F(\alpha, \psi, \vec{p}) := 0$.
O/w, set $F(\alpha, \psi, \vec{p}) := \min\{\rho_Z(w) ; Z \models \psi(w, \vec{p})\}$

If $\psi \in \Phi$, define

$$F_\psi(\alpha) := \sup\underbrace{\{F(\alpha, \psi, \vec{p}); \vec{p} \in Z_\alpha^{<\omega}\}}$$

$\{F(\alpha, \psi, \vec{p}); \vec{p} \in Z_\alpha^{<\omega}\}$ is a set because of Replacement in M

So, the supremum exists.

Define $F(\alpha) := \max\{F_\psi(\alpha); \psi \in \Phi\} \cup \{\alpha+1\}$

to make it strictly increasing

$$\vartheta_0 := \alpha$$

$$\vartheta_{i+1} := F(\vartheta_i)$$

$$\vartheta := \sup\{\vartheta_i; i \in \omega\}$$

Since $\vartheta_0 < \vartheta_1 < \vartheta_2 < \dots$, ϑ is a limit ordinal.

<u>Claim</u> φ is absolute between Z_ϑ and Z:

[Show that all fmlas in Φ are abs. between Z_ϑ & Z.]

By improved TVL, only need to show the TV condition.

Suppose $Z \models \exists x\, \psi(x, \vec{p})$ where $\vec{p} \in Z_\vartheta^{<\omega}$

Every $p_i \in Z_\alpha$ with $\alpha < \vartheta$

So there is $\bar{\alpha} < \vartheta$ s.t. $\vec{p} \in Z_{\bar{\alpha}}^{<\omega}$.

There is some i s.t. $\vartheta_i \leq \bar{\alpha} < \vartheta_{i+1}$

Thus $\vec{p} \in Z_{\vartheta_{i+1}}^{<\omega}$.

Therefore by definition of ϑ_{i+2}, there is $a \in Z_{\vartheta_{i+2}} \subseteq Z_\vartheta$ s.t.

$$Z \models \psi(a, \vec{p}).$$

That's what TVL needs us to show.

q.e.d.

Summary For every hierarchy and every single formula φ, I can find a sufficiently large ϑ s.t.

$$\{y \in x;\ Z \models \varphi(y, \vec{p})\}$$

$$= \{y \in x;\ Z_\vartheta \models \varphi(y, \vec{p})\}.$$

MONDAY 27 May 2024 due to Pentecost

In Lecture VII, we'll use this to fix the gap in our attempt to prove Separation in L and also prove Replacement.

[Also discuss $L \models AC$ which is a relative consistency proof of ZFC from ZF.]