The Constructible Universe

Fifth Lecture
6 May 2024

RECAP

OVERALL GOAL:
Start with $N \models ZF$ and construct an inner model $M \subseteq N$ s.t. $M \models ZFC + CH$.

Could do ZFC, but our method gives us the consistency of AC for free.

QUESTION
How do we control the truth value of statements in inner models.

ABSOLUTENESS:
For transitive models, Δ_0 formulas are absolute, Σ_1 formulas are upwards absolute, Π_1 formulas are downwards absolute, Δ_1 formulas are absolute.

IMPORTANT
In general, being a cardinal is not absolute.

ORAL EXAMS FOR C.U.

between 22 & 30 July 2024
[not prerequisites]

Lecture IV

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Theorem Suppose M is transitive in N.

(i) If $\mathcal{P}(\omega)^M = \mathcal{P}(\omega)^N$, then $\aleph_1^M = \aleph_1^N$.

This notation means: the formula defining $\mathcal{P}(\omega)$

Φ [i.e., the unique object that consists of all subsets of the unique object that is the smallest inductive set

relativised to M or N, resp., i.e. Φ^M or Φ^N

(ii) If $\mathcal{P}(\omega)^M = \mathcal{P}(\omega)^N$, then $N \models \neg CH \Longrightarrow M \models \neg CH$.

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Theorem If $M \subseteq N$ is transitive s.t. "x is a cardinal" is absolute between M and N.

Then $N \models CH \Longrightarrow M \models CH$.

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Corollary If $M \subseteq N$ and $\mathcal{P}(\omega)^M = \mathcal{P}(\omega)^N$ and "x is a cardinal" is absolute between M & N

then $M \models CH \Longleftrightarrow N \models CH$.

If M contains all subsets of $\mathbb{N}$, then CH is upwards absolute.

If "x is a cardinal" is absolute, then CH is downwards absolute. Not important for us: important for the FORCING context.

Consequence If we wish to change the value of CH, we must remove subsets of $\mathbb{N}$ or destroy cardinals.

§ 10 Defining Definability

Motivation. Why "definability"?

Heuristics: All definable subsets of $\mathbb{N}$ must be in the inner model M, so constructing an inner model M s.t.

$$\mathcal{P}(\omega)^M = \text{all definable subsets of } \omega$$

would be the minimal choice.

Note that there are only ctbly many definable (WITHOUT PARAMETER) subsets of $\mathbb{N}$ and there is a surjection from $\mathbb{N}$ onto that set, so that can't be $\mathcal{P}(\mathbb{N})$.

So, we rather mean "definable w/ parameters".

Let's be more precise

Fix $N \models ZF$ (if you like, take $N = V_\lambda$ for λ inaccessible)

Say $A \subseteq N$ is definable if there is φ formula s.t. $x \in A \longleftrightarrow N \models \varphi(x)$.

Two things not necessarily expressible in N:

(i) "there is a formula φ".

(ii) "$N \models \ldots$".

[Remark: (ii) relates to Tarski's Undefinability of Truth Theorem.]

Say that $A \subseteq N$ is definable from parameters $p_0, \dots, p_n \in N$

if there is φ formula s.t.

$$x \in A \longleftrightarrow N \models \varphi(x, p_0, \dots, p_n).$$

Remark. Since our models are models of set theory and $N^{<\omega} \subseteq N$ [in other words: if $p_0, \dots, p_n \in N$, then so is $(p_0, \dots, p_n) \in N$.] we can always use a single parameter, i.e.,

$A \subseteq N$ is definable from parameters if there is φ formula and $p \in N$ s.t. $x \in A \longleftrightarrow N \models \varphi(x, p)$.

Back to our problems: (i) & (ii).

(ii) is the more serious one:

We CANNOT define definability.

This means: there is no formula Δ s.t. for all $x \in N$ we have

$$N \models \Delta(x) \Longleftrightarrow x \text{ is definable}$$

$$\Longleftrightarrow \text{there is } \varphi \text{ s.t. } z \in x \longleftrightarrow N \models \varphi(z).$$

Proof. Suppose such a Δ exists.

Write

$\Phi(\alpha) :\Leftrightarrow$ α is an ordinal and

$\forall \beta < \alpha \ \ \Delta(\beta)$ and

$\neg \Delta(\alpha)$.

In words: "α is the least non-definable ordinal".

But Φ witnesses that this is definable.

Contradiction!

q.e.d.

We will use that even though definability in N cannot be expressed in N, definability over other structures, i.e., $(X, \in)$ s.t. $X \in N$, can be defined in N.

Back to problem (i): formulas are syntactic objects and not elements of N.

While fmlas are not elements of, we can identify them with elements of $\omega^{<\omega}$:

Let Σ be the alphabet of our language:

$\in$	$=$	$\wedge$	$\neg$	$\exists$	(	)	v_0	v_1	v_2	v_3	...
0	1	2	3	4	5	6	7	8	9	10	

$\mathrm{Var} := \{i+7 ; i \in \mathbb{N}\}$ $\qquad v_i \leadsto i+7$

E.g., $\exists v_0 \exists v_1 \; v_0 = v_1$

4 7 4 8 7 1 8

$(4,7,4,8,7,1,8)$ codes the fmla $\exists v_0 \exists v_1 v_0 = v_1$

Write: $\text{code}(\exists v_0 \exists v_1 v_0 = v_1)$.

Defining well-formed fmlas from this coding:

Usually by recursion:

$(i+7, 0, j+7)$

$$\text{Fml}(0, \varphi) = 1 \;:\Longleftrightarrow\; \varphi \in \omega^{<\omega} \wedge \exists i,j \in \omega \big(\varphi = \text{code}(v_i \in v_j) \vee \varphi = \text{code}(v_i = v_j) \big)$$

$(i+7, 1, j+7)$

$$\text{Fml}(i+1, \varphi) = 1 \;:\Longleftrightarrow\; \text{Fml}(i, \varphi) = 1 \;\vee\; \exists \psi, \psi' \big(\text{Fml}(i, \psi) = 1 \wedge \text{Fml}(i, \psi') = 1 \wedge (\varphi = \neg \psi \vee \varphi = \psi \wedge \psi' \vee \varphi = \exists v_i \psi) \big)$$

Since $N \models ZF$, the Recursion Theorem holds in N, so this recursion defines a unique function and we can define

$$\text{Fml}(i) := \{\varphi ; \text{Fml}(i, \varphi) = 1\}$$

$$\text{Fml} := \bigcup_{i \in N} \text{Fml}(i)$$

Similarly, define by recursion whether a variable occurs **freely** in a formula and by this

$$\mathrm{Fml}_k := \{\varphi \in \mathrm{Fml};\ \varphi \text{ has free variables } v_0, \dots, v_{k-1}\}$$

Thus Fml_0 is the sentences.

Q. Is this recursive definition absolute?

THEOREM (Transfinite) Recursion preserves absoluteness.

I.e., if F is given by recursion equations

$$F(0,x) = G_1(x)$$
$$F(\alpha+1,x) = G_2(F(\alpha,x),x)$$
$$F(\lambda,x) = G_3(\{F(\alpha,x);\alpha<\lambda\},x) \quad \text{if } \lambda \text{ limit}$$

and G_1, G_2, G_3 are all absolute between M & N, then so is F.

Proof. Remember the proof of the Recursion Theorem:

g is called an x-germ if $\mathrm{dom}(g) \in \mathrm{Ord}$ and for all $\beta \in \mathrm{dom}(g)$

$$g(\beta) = G_1(x) \quad \text{if } \beta = 0$$
$$g(\beta) = G_2(g(\alpha),x) \quad \text{if } \beta = \alpha+1$$
$$g(\beta) = G_3(g \upharpoonright \beta, x) \quad \text{if } \beta \text{ is limit}$$

..

In ZF, we prove that

(1) for each α there is x-germ g s.t. $\alpha \in \mathrm{dom}(g)$

(2) if $\alpha \in \mathrm{dom}(g) \cap \mathrm{dom}(g')$ for two x-germs g, g', then $g(\alpha) = g'(\alpha)$.

Define $F(\alpha, x) := g(\alpha)$ for any x-germ g s.t. $\alpha \in \mathrm{dom}(g)$

Since G_1, G_2, G_3 are absolute, being an x-germ is an absolute property.

But since (1) & (2) are ZF-theorems and therefore true in both M & N and the resulting germs used in the definition of F in M and N are germs in the other model, they have to agree by (2) and therefore F is absolute.

q.e.d.

Remark This is formulated for transfinite recursion; the case of standard recursions on $\mathbb{N}$ is a special case.

Corollary Being a formula, being a formula of rank i, being a formula with k free variables are all absolute between trs models of ZF.

Next goal Define "$X \models \dots$" for $X \in N$.

Idea. The standard Tarski definition of validity is a recursive definition. So if it only uses absolute notions, this will make definability over a structure absolute.

stands either for "truth" or "Tarski"...

$$T(0, X, \varphi, I) = 1 :\Longleftrightarrow \varphi \in Fml(0) \wedge I : Var \longrightarrow X$$
$$\wedge \big(\varphi = v_i \in v_j \wedge I(i+7) \in I(j+7)$$
$$\vee \ \varphi = v_i = v_j \wedge I(i+7) = I(j+7) \big)$$

similar for $= 0$

$$T(i+1, X, \varphi, I) = 1 \Longleftrightarrow \varphi \in Fml(i) \wedge T(i, X, \varphi, I) = 1$$
$$\vee \ \exists \psi, \psi' \in Fml(i)$$
$$\big[\ \varphi = \psi \wedge \psi' \wedge T(i, X, \psi, I) = 1 \wedge T(i, X, \psi', I) = 1$$
$$\vee \ \varphi = \neg \psi \wedge T(i, X, \psi, I) = 0$$
$$\vee \ \varphi = \exists v_k \psi \wedge \exists J : Var \longrightarrow X \ \forall m \in \omega \ m \neq k \longrightarrow J(m+7) = I(m+7)$$
$$\wedge \ T(i, X, \psi, J) = 1 \ \big]$$

similarly for $= 0$

.

Note that the G_2-formula for the successor step involves quantifiers:

$\exists \varphi, \varphi' \in \mathrm{Fml}(i)$

$\exists \mathcal{I} : \mathrm{Var} \longrightarrow X$

$\forall m \in \omega$

Since the models are models of set theory and $\mathrm{Fml}(i)$ and Var and $\{f ; f : \mathrm{Var} \longrightarrow X\}$ are absolute, all of these quantifiers are bounded in N.

IMPORTANT. This uses that $X \in N$, otherwise the quantifier $\exists \mathcal{I} : \mathrm{Var} \longrightarrow X$ would be unbounded !!!

Corollary If $M \subseteq N$ are models of ZF and $X \in M$, then "$X \models \varphi$" is absolute between M and N.

Therefore, there is a formula that defines definability from parameters over X absolutely between M & N.

$$\Delta(a, X) :\Longleftrightarrow \exists \varphi \exists p \forall x \; \left(x \in a \longleftrightarrow X \models \varphi(x, p) \right)$$

where $X \models \varphi(x,p)$: $\exists i \, \exists \mathcal{I} : \mathrm{Var} \longrightarrow X$ with p, $T(i, X, \varphi, \mathcal{I}) = 1$ and $\mathcal{I}$ is consistent

§ 11 Definition of the constructible universe.

Definition If X is a set, then let

$$\mathcal{D}(X) :=$$
$$\{A \subseteq X;\ A \text{ is definable with parameters over } X\}$$
$$= \{A \subseteq X;\ N \models \Delta(A, X)\}$$

Sometimes called "the definable power set operation".

Reminder
$V_0 := \emptyset$
$V_{\alpha+1} := \wp(V_\alpha)$
$V_\lambda := \bigcup_{\alpha<\lambda} V_\alpha$
von Neumann hierarchy
[Forming this within N produces N since N ⊨ Foundation.]

Observe ① For all X, $\mathcal{D}(X) \subseteq \wp(X)$.

② If X is transitive, then so is $\mathcal{D}(X)$.

[If $a \in \underbrace{b \in \mathcal{D}(X)}$. But every element of X is definable over X with parameters.
$\Rightarrow b \subseteq X$
$\Rightarrow a \in X$
$\Longrightarrow a \in \mathcal{D}(X)$.]

The constructible hierarchy

$$L_0 := \emptyset$$
$$L_{\alpha+1} := \mathcal{D}(L_\alpha)$$
$$L_\lambda := \bigcup_{\alpha<\lambda} L_\alpha$$

History of letter L unknown.

Note that the $\mathcal{D}$-operator was shown to be absolute in § 10.

Main Theorem of § 10 said that transfinite recursion preserves absoluteness.

Corollary The statement

$$x \in L_\alpha$$

is absolute for transitive models of set theory.

Definition x is called constructible if there is $\alpha \in \mathrm{Ord}$ s.t. $x \in L_\alpha$.

Corollary If $M \subseteq N$ and $\mathrm{Ord} \cap M = \mathrm{Ord} \cap N$, then "$x$ is constructible" is absolute between M & N.

Thus if I write $L := \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha$, then L (the constructible sets) is contained in every model of ZF.

Thus. If we can show that $L \models ZF$, then L is the MINIMAL MODEL of ZF.