# IX Ninth Lecture of THE CONSTRUCTIBLE UNIVERSE

10 June 2024

NOTE THAT THE LECTURE NOTES OF LECTURES VII & VIII HAVE BEEN UPDATED.

## Lecture VIII

### §16 The condensation sentence

Let
$$\sigma^* := \bigwedge T_{\Phi} \wedge \bigwedge T_{\Phi^*} \wedge \bigwedge T_{\mathrm{ord}} \wedge \mathrm{Found} \wedge V = \mathbb{L}$$

We now call $\sigma^*$ the CONDENSATION SENTENCE

(Lecture VII, proved in Lecture VIII)

**Theorem** If $X$ is transitive, $X \models \sigma^*$, and $\alpha := X \cap \mathrm{Ord}$, then $X = \mathbb{L}_\alpha$.

The proof will be given in Lecture VIII.

$\sigma^*$ (finitely) axiomatizes "being an $L_\alpha$" for transitive sets.

Q (Mucke): Is there a similar formula for the V-hierarchy?

### §17 Constructible ranks & inaccessibles

Lecture VIII, p. 9

**Theorem** If $\kappa$ is inaccessible, then there is a $\alpha < \omega_1$ s.t. $L_\alpha \models \mathrm{ZFC}$.

Lecture VIII, p. 10

Remark In the "V-hierarchy" version of this proof you have to define $H^+_{\mathrm{un}}(A) := V_\beta$ where $\beta$ is least s.t. all elts of $H_{\mathrm{un}}(A)$ are in $V_\beta$. Then $\mathcal{H}^+(A) = V_\gamma$ for some $\gamma$, but we lose control over cardinality. In general $\gamma$ is a cardinal and quite big.

Before we proceed to CH, let's see that there is no finite axiomatisation of "being a $V_\alpha$".

<u>Reminder</u> $Z_\alpha$ (for $\alpha \in \mathrm{Ord}$) was a <u>hierarchy</u> if

1. $Z_\alpha$ is transitive
2. $\mathrm{Ord} \cap Z_\alpha = \alpha$
3. $\alpha < \beta \Longrightarrow Z_\alpha \subseteq Z_\beta$
4. $\lambda$ limit $\Rightarrow Z_\lambda = \bigcup_{\alpha<\lambda} Z_\alpha$

<u>Def.</u> A hierarchy $Z$ is finitely axiomatisable (for trs sets) if there is a sentence $\tau$ s.t. f.a. $X$ transitive

$$X \models \tau \Longrightarrow \exists \alpha \; X = Z_\alpha .$$

In §16, we proved that the $L$-hierarchy is finitely axiomatisable ($\tau = \sigma^*$).

In §17, we proved:

<u>Theorem</u> If $Z$ is finitely axiomatisable and there is $\kappa$ s.t. $Z_\kappa \models \mathrm{ZFC}$, then there is $\alpha < \omega_1$ s.t. $Z_\alpha \models \mathrm{ZFC}$.

Sketch of the proof of that Theorem (done in L $\overline{\text{VIII}}$ in the case of $Z = \mathbb{L}$):

1. Assume $Z_\kappa \models ZFC$.
2. Build $\mathcal{H}^{Z_\kappa}(\emptyset) \prec Z_\kappa$ countable.
3. Note that by finite axiomatisability, we $Z_\kappa \models \tau$, so $\mathcal{H}^{Z_\kappa}(\emptyset) \models ZFC + \tau$.
4. Let $X$ be the Mostowski collapse of $\mathcal{H}^{Z_\kappa}(\emptyset)$, also countable:
$$X \models ZFC + \tau.$$
5. By choice of $\tau$ $\exists \alpha$ $X = Z_\alpha$. By countability (and $Z_\alpha \cap \mathrm{Ord} = \alpha$) get $\alpha < \omega_1$.

Mudee's question from Lecture $\overline{\text{VIII}}$:
Is the V-hierarchy finitely axiomatisable.

<u>Proposition</u> If $V_\alpha \models ZFC$, then $\alpha$ is a cardinal.

[In particular, $\alpha \geq \omega_1$.]

<u>Corollary</u> The V-hierarchy is not finitely axiomatisable.
[Modulo the existence of $\kappa$ s.t. $V_\kappa \models ZFC$.]

Proof of Proposition

Assume $V_\alpha \models ZFC$ and $\alpha$ is not a cardinal.

$\boxed{|\alpha| = \kappa}$ for some cardinal $\kappa < \alpha$.

Clearly $\alpha$ is a limit ordinal.

[ZFC $\vdash$ there is no largest ordinal]

$|\alpha| = \kappa$ means that there is $E \subseteq \kappa \times \kappa$,

s.t. $(\kappa, E) \cong (\alpha, \in)$.

What's the rank of $E$?

$(\alpha, \beta) \in E$

$= \{\{\alpha\}, \{\alpha, \beta\}\}$

$\Longrightarrow E \subseteq V_\kappa$

$\Longrightarrow E \in V_{\kappa+1} \subseteq V_\alpha$.

[since $\alpha$ was limit ordinal]

So $V_\alpha$ contains an object that is isomorphic to $\alpha$.

Since $V_\alpha \models ZFC \Longrightarrow$

$V_\alpha \models$ there is an ordinal isomorphic to $(\kappa, E)$.

$\Longrightarrow \alpha \in V_\alpha$ ↯ q.e.d.

However, there is V-version of the argument for the theorem from Lecture VIII.

[Reminder We say $\kappa$ has **countable cofinality**
if there are $\kappa_i < \kappa$ (for $i \in \mathbb{N}$) s.t.
$$\kappa = \bigcup_{i \in \mathbb{N}} \kappa_i .$$
[Example: $\aleph_\omega = \bigcup_{i \in \mathbb{N}} \aleph_i$ . ]

**Theorem** If $V_\kappa \models ZFC$ then there is $\alpha \leq \kappa$ of countable cofinality s.t. $V_\alpha \models ZFC$.

In particular, if $\kappa$ has uncountable cofinality (e.g., an uncountable regular cardinal), then $\alpha < \kappa$.

**Proof.** We follow the proof sketch of page 3, except that we have finite axiomatisation $\tau$, so we need to make sure that the model is a $V_\alpha$ in a different way.

SKOLEM HULL of A

**Remember**
$$H_0(A) := A$$
$$H_{i+1}(A) := \text{Witness}(H_i(A))$$
$$\mathcal{H}(A) := \bigcup_{i \in \mathbb{N}} H_i(A)$$

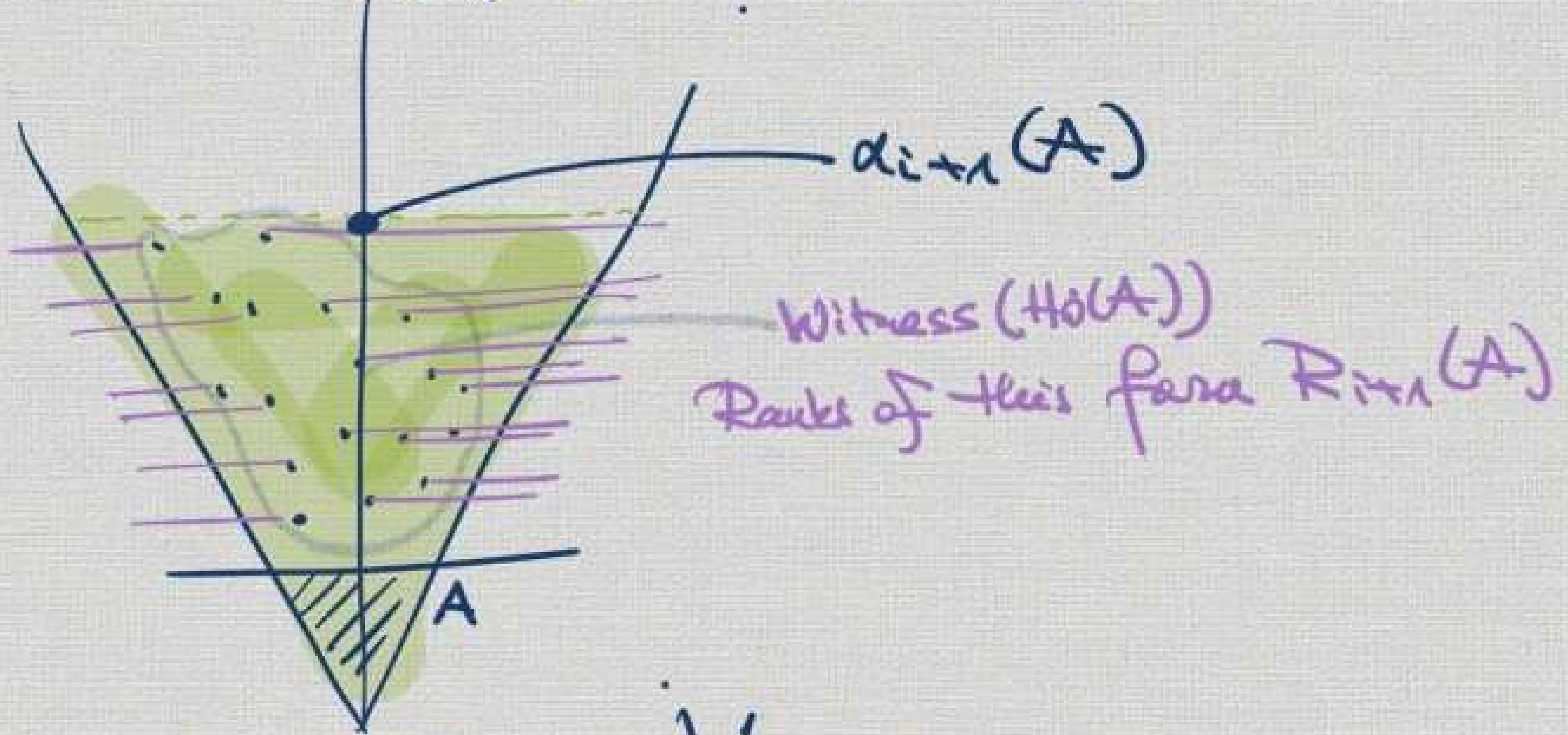
set of selected witnesses of $\exists$-formulas with parameters in $H_i(A)$

Instead, we define

$$R_{i+1}(A) := \{\rho(x);\ x \in H_{i+1}(A)\}$$

$H_{i+1}(A) := \mathrm{Witness}(H_i(A))$

$$\alpha_{i+1}(A) := \sup R_{i+1}(A)$$




$$H_{i+1}(A) := V_{\alpha_{i+1}(A)}.$$

With this definition and $\mathcal{H}(A) := \bigcup_{i \in \mathbb{N}} H_i(A)$.

we have $\mathcal{H}(A) = V_\alpha$

where $\alpha = \sup\{\alpha_i(A);\ i \in \mathbb{N}\}$ (*)

By Tarski-Vaught $\mathcal{H}(A) \preccurlyeq V_\kappa$, so

$\mathcal{H}(A) = V_\alpha \models ZFC$.

By (*), $\alpha$ has countable cofinality.

q.e.d.

<u>Remark</u> Since $H_0(\emptyset)=\emptyset$, then
$\omega_1, \omega_2, \omega_3, \ldots \in \text{Witness}(H_0(\emptyset))$,
so $\alpha_1(\emptyset) \geq \aleph_\omega$ and for bigger,
so $H_1(\emptyset) \supseteq V_{\aleph_\omega}$ and therefore –
this hierarchy of sets is growing
very quickly.

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## §18 CH in the constructible universe

What do we know about the $\mathbb{L}$-powerset of $\mathbb{N}$?
When we proved that

$$\mathbb{L} \models \text{Powerset},$$

we proved that $\wp(\mathbb{N})$ exists.
We had observed that by Replacement,
there has to be some $\gamma$ s.t.

$$\wp(\mathbb{N}) \cap \mathbb{L} \subseteq \mathbb{L}_\gamma.$$

We had no bound on the size of $\gamma$.

If $V=L$, then $\wp(\mathbb{N}) \cap \mathbb{L}$ is uncountable, so
for any $\alpha<\omega_1$, we have $|\mathbb{L}_\alpha| < \aleph_1$,
so $\mathbb{L}_\alpha \cap \wp(\mathbb{N}) \subsetneq \mathbb{L} \cap \wp(\mathbb{N})$.

So, there are two possible scenarios:

I. there is $\alpha \geq \omega_1$ s.t.
$$\rho_{\mathbb{L}}'(A) = \alpha \text{ for some } A \subseteq \mathbb{N}$$

II. there are unboundedly many $\alpha < \omega_1$ s.t. $\rho_{\mathbb{L}}(A) = \alpha$ for some $A \subseteq \mathbb{N}$.

**Remark** I. + II. together could be possible.

If $\neg$ I., then II. and in this situation
$$\wp(\mathbb{N}) \cap \mathbb{L} \subseteq \mathbb{L}_{\omega_1}.$$
$$|\wp(\mathbb{N}) \cap \mathbb{L}| \leq |\mathbb{L}_{\omega_1}| = \aleph_1.$$

So if we can show that I. does not apply, we have proved that if $V=L$
$$2^{\aleph_0} = |\wp(\mathbb{N})| \leq \aleph_1.$$

I.e., CH.

**Theorem** (THE CONDENSATION LEMMA)

Assume $V=L$.

If $A \subseteq \mathbb{N}$, then there is $\alpha < \omega_1$ s.t. $A \in L_\alpha$.

**Corollary** $V=L \Longrightarrow CH$.

[As argued before.]

**Proof of the Condensation Lemma**

We would like to take a Skolem hull of $\mathbb{L}$, but we can only take Skolem hulls inside sets. This would work nicely if there was a $\kappa$ s.t. $\mathbb{L}_\kappa \models ZFC$, but we cannot assume that; but this assumes that there is an inaccessible cardinal (whose existence is unprovable in ZFC).

Since $\sigma^*$ (the condensation sentence) is a finite conjunction of axioms of ZFC and $V=L$, we know that

$$\mathbb{L} \models \sigma^*.$$

So, by Lévy Reflection Theorem, there are unboundedly many $\vartheta$ s.t. $\mathbb{L}_\vartheta \models \sigma^*$.

Fix $A \subseteq \mathbb{N}$ and $\gamma$ s.t. $A \in \mathbb{L}_\gamma$.

By LRT, find $\vartheta > \alpha$ s.t. $A \in \mathbb{L}_\vartheta \models \sigma^*$.

Do the Skolem hull construction inside $\mathbb{L}_\vartheta$, i.e., form

$$\mathcal{H}^{\mathbb{L}_\vartheta}(\{A\}) \preccurlyeq \mathbb{L}_\vartheta .$$

In particular,

① since $\{A\}$ is finite, $\mathcal{H}^{\mathbb{L}_\vartheta}(\{A\})$ is countable

② since $\mathbb{L}_\vartheta \models \sigma^*$, $\mathcal{H}^{\mathbb{L}_\vartheta}(\{A\}) \models \sigma^*$.

Form Mostowski collapse of $\mathcal{H}^{\mathbb{L}_\vartheta}(\{A\})$ and get transitive set $X$ isomorphic to $\mathcal{H}^{\mathbb{L}_\vartheta}(\{A\})$.

Since it's isomorphic, $X$ is countable and $X \models \sigma^*$.

Thus there is some $\alpha$ s.t. $X = \mathbb{L}_\alpha$, so $\alpha < \omega_1$ by countability.

In order to see that $A \in X$, we note that the Mostowski isomorphism is the identity on transitive sets. Note

assuming that $A$ is definite

$$\mathrm{tcl}(\{A\}) = \{A\} \cup \mathbb{N}$$

$$[\text{since } A \subseteq \mathbb{N}]$$

Thus the Mostowski iso is the identity on $\{A\} \cup \mathbb{N}$ and therefore maps $A$ to $A$, thus $A \in X = \mathrm{ran}(\pi)$, where $\pi$ is the Mostowski iso.

q.e.d.

## § 19 Cardinal arithmetic in II.

CH: $\quad 2^{\aleph_0} = \aleph_1.$

Cantor's Theorem: $\quad 2^{\kappa} \geq \kappa^{+}.$

Generalised Continuum Hypothesis:

For all $\kappa$, $\quad 2^{\kappa} = \kappa^{+}.$

GCH

**Theorem** $V = L \Longrightarrow GCH$.

**Remark** This is exactly the same argument as in the CH case.

**Proof** Fix $\kappa$. By $\mathbb{L} \models$ Powerset, we know that there is $\gamma > \kappa$ s.t.

$$\mathcal{P}(\kappa) \cap \mathbb{L} \in \mathbb{L}_\gamma.$$

As before, if $\boxed{\forall A \subseteq \kappa \ \exists \alpha < \kappa^+ \text{ s.t. } A \in \mathbb{L}_\alpha}$, then (*)

$$\mathcal{P}(\kappa) \cap \mathbb{L} \subseteq \mathbb{L}_{\kappa^+}$$

$$\left.\begin{array}{l} \mathbb{L} \models 2^\kappa \leq \kappa^+. \\ ZFC \vdash 2^\kappa \geq \kappa^+ \end{array}\right\} \Rightarrow \mathbb{L} \models 2^\kappa = \kappa^+.$$

(*) is the CONDENSATION LEMMA.

Let's prove (*).

As before, find $\vartheta > \gamma$ s.t.

$$A \in \mathbb{L}_\vartheta \models \sigma^*$$ (the condensation sentence)

Now form inside $\mathbb{L}_\vartheta$ a Skolem hull.

Consider $tcl(\{A\}) = \{A\} \cup \kappa = A^*$. [if $A$ is unbounded in $\kappa$]

Note $A^*$ is transitive and has size $\kappa$.

Form
$$\mathcal{H}^{\mathbb{L}_\vartheta}(A^*) \prec \mathbb{L}_\vartheta .$$
As before, get

① $|\mathcal{H}^{\mathbb{L}_\vartheta}(A^*)| = |A^*| = \kappa$

② $\mathcal{H}^{\mathbb{L}_\vartheta}(A^*) \models \sigma^*$.

Form the Mostowski collapse of $\mathcal{H}^{\mathbb{L}_\vartheta}(A^*)$, say $X$. That is

⓪ transitive

① $|X| = \kappa$

② $X \models \sigma^*$.

So, by §16, we get that $X = L_\alpha$ with $|L_\alpha| = \kappa \implies \alpha < \kappa^+$.

Observe that $A \in X$ by the same argument as before: $A^*$ is transitive and therefore the Mostowski collapse is the identity on $A^*$ and therefore preserves $A$.

q.e.d.

Remark: If GCH holds, all of cardinal arithmetic is mostly determined.

Example: $\aleph_7^{\aleph_2} = ?$

The Hausdorff formula is

$$\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1}$$

Thus:
$$\begin{aligned}
\aleph_7^{\aleph_2} &= \aleph_6^{\aleph_2} \cdot \aleph_7 = \aleph_5^{\aleph_2} \cdot \aleph_6 \cdot \aleph_7 \\
&= \aleph_4^{\aleph_2} \cdot \aleph_5 \cdot \aleph_6 \cdot \aleph_7 \\
&= \aleph_3^{\aleph_2} \cdot \aleph_4 \cdot \aleph_5 \cdot \aleph_6 \cdot \aleph_7 \\
&= \underbrace{\aleph_2^{\aleph_2}} \cdot \aleph_3 \cdot \aleph_4 \cdot \aleph_5 \cdot \aleph_6 \cdot \aleph_7 \\
&= \aleph_3 \cdot \aleph_7 = \aleph_7 .
\end{aligned}$$

As a consequence, if $V = L$, cardinal arithmetic is as simple as it can be.

Remaining four lectures

1. $\diamondsuit$ ("diamond") in $L$. [Strengthening of CH.]
2. Descriptive set theory in $L$: descriptive complexity of nasty sets.