# The Constructible Universe IV

Fourth Lecture 29 April 2024

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Recap.

Absolute for transitive models:

Function, Injection, Surjection, Subset,

Ordinal* (only if Foundation is true)

Downwards absolute for transitive models:

Cardinal $\Pi_1$, downwards absolute

Q. Is being a cardinal absolute?

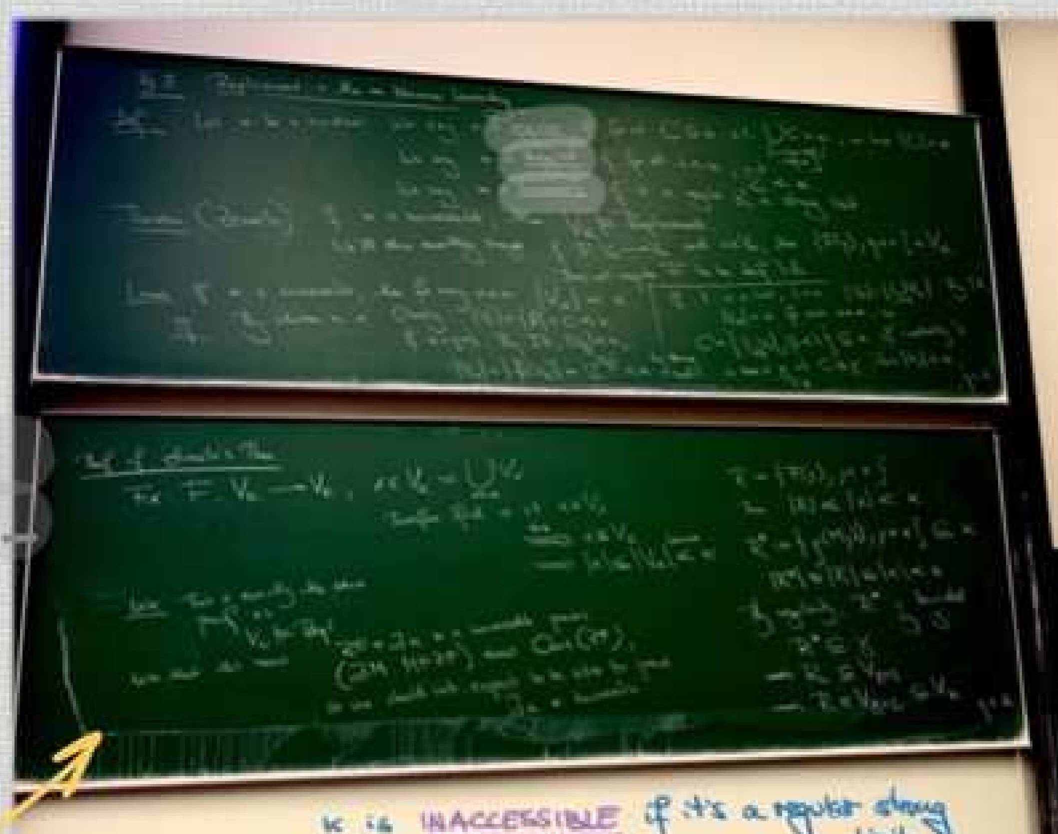
Lecture III

If $\alpha > \omega$ limit, then

$V_\alpha \models Z$

(Zermelo set theory: everything w/o Replacement)

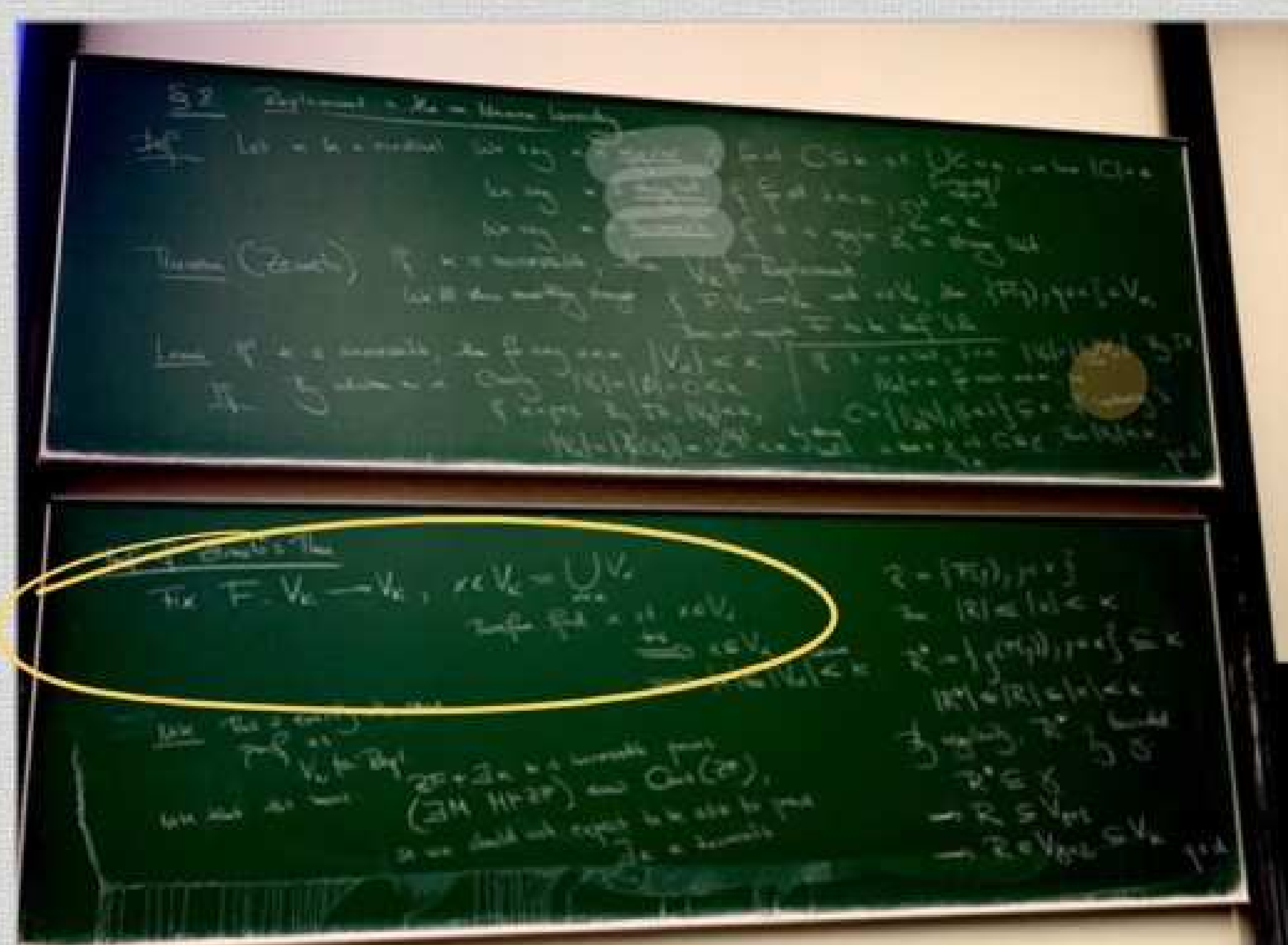
If $\kappa$ is inaccessible, then $V_\kappa \models ZF$.



$\kappa$ is INACCESSIBLE if it's a regular strong limit

Lemma If $\kappa$ is inaccessible, then $\forall \alpha < \kappa \; |V_\alpha| < \kappa$.

Theorem If $\kappa$ is inaccessible, then $V_\kappa \models$ Replacement.



$\kappa$ is INACCESSIBLE if it's a regular strong limit

We proved something stronger:

We proved that for every fn $F: V_\kappa \to V_\kappa$, the range of $F$ restricted to a set is a set.

Lemma If $\kappa$ is inaccessible, then $\forall \alpha < \kappa$ $|V_\alpha| < \kappa$.

Theorem If $\kappa$ is inaccessible, then $V_\kappa \models$ Replacement.

Q. Is this theorem an equivalence, i.e., if $V_\kappa \models$ Replacement, then $\kappa$ is inaccessible.

A. NO (later today)

Replacement would only have definable functions here.

How many are these:

DEFINABLE w/o PARAMETER: $\aleph_0$

DEFINABLE w/ PARAMETERS FROM $V_\kappa$: $|V_\kappa| = \kappa$.

However, there are $2^\kappa$ many functions from $V_\kappa$ to $V_\kappa$

**Remark** There is a converse of Zermelo's Thm, but not with Replacement but rather the stronger statement we proved in the proof.

**Def.** $V_\kappa$ satisfies SECOND-ORDER REPLACEMENT (SOR) if f.a. $F : V_\kappa \longrightarrow V_\kappa$ and all $x \in V_\kappa$ $\{F(y) \mid y \in x\} \in V_\kappa$.

**Thm** (SHEPHERDSON'S THEOREM)
If $V_\kappa$ satisfies SOR, then $\kappa$ is inaccessible.

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## §9 Countable transitive submodels.

For reasons of simplicity, let's assume that $\kappa$ is inaccessible and thus $V_\kappa \models ZF$.
[This assumption is an overkill & only for convenience.]

Note that Löwenheim-Skolem tells us that if ZF is consistent, it has a countable model.

Sometimes known as "Skolem paradox", but there is nothing paradoxical about it.

Our goal: a transitive model of ZF.

## Model-theoretic technique

M ≼ N elementary substructure if all formulas are absolute.

### Tarski-Vaught Lemma
(a.k.a. Tarski-Vaught Test)

$M \preccurlyeq N \iff$
for every formula $\varphi$ and $m_1, \dots, m_n \in M$, we have
if $N \models \exists x\, \varphi(x, m_1, \dots, m_n)$,
there is $m \in M$ s.t.
$N \models \varphi(m, m_1, \dots, m_n)$.

Proof. $\Longrightarrow$ is obvious.
$\Longleftarrow$ is a proof by induction on the complexity of formulas.

We have already seen:
(1) Atomic fmlas are absolute.
(2) Absolute fmlas are closed under propositional connectives.

So, only to show: if $\varphi$ is absolute, then so is $\exists x \varphi$.

$M \models \exists x \varphi(x, m_1, \dots, m_n) \iff N \models \exists x \varphi(x, m_1, \dots, m_n)$

$\Longrightarrow$ is trivial since $M \subseteq N$

$\Longleftarrow$ if $N \models \exists x \varphi(x, m_1, \dots, m_n)$, then by the TV condition get $m \in M$ s.t.
$N \models \varphi(m, m_1, \dots, m_n)$
By IH, $M \models \varphi(m, m_1, \dots, m_n)$. So
$M \models \exists x\, \varphi(x, m_1, \dots, m_n)$. q.e.d.

Let's apply that to $V_\kappa \models ZF$.

$$M_0 := \emptyset.$$

If $M_i$ is already defined, look at all fmlas $\varphi$ and all $\vec{p} \in M_i^n$ and consider,

n+1 free variables

$$V_\kappa \models \exists x\, \varphi(x, \vec{p})$$

if false, let

$$w(\varphi, \vec{p}) := \emptyset$$

if true, there is some $x \in V_\kappa$ s.t. $V_\kappa \models \varphi(x, \vec{p})$

So pick [uses AC] some $w(\varphi, \vec{p})$ s.t.

$$V_\kappa \models \varphi(\underline{w(\varphi, \vec{p})}, \vec{p}).$$

$$M_{i+1} := M_i \cup \{ w(\varphi, \vec{p});\ \varphi \text{ formula}, \vec{p} \in M_i^{<\omega} \}$$

Finally, let $M := \bigcup_{i \in \mathbb{N}} M_i$.

<u>Claim</u> $M \preccurlyeq V_\kappa$.

By TV, I only have to show if $V_\kappa \models \exists x\, \varphi(x, \vec{p})$ with $\vec{p} \in M^{<\omega}$ then there is $m \in M$ $V_\kappa \models \varphi(m, \vec{p})$.

Since $\vec{p} \in M^{<\omega}$, we find $k_i$ s.t.

$$p_i \in M_{k_i},$$

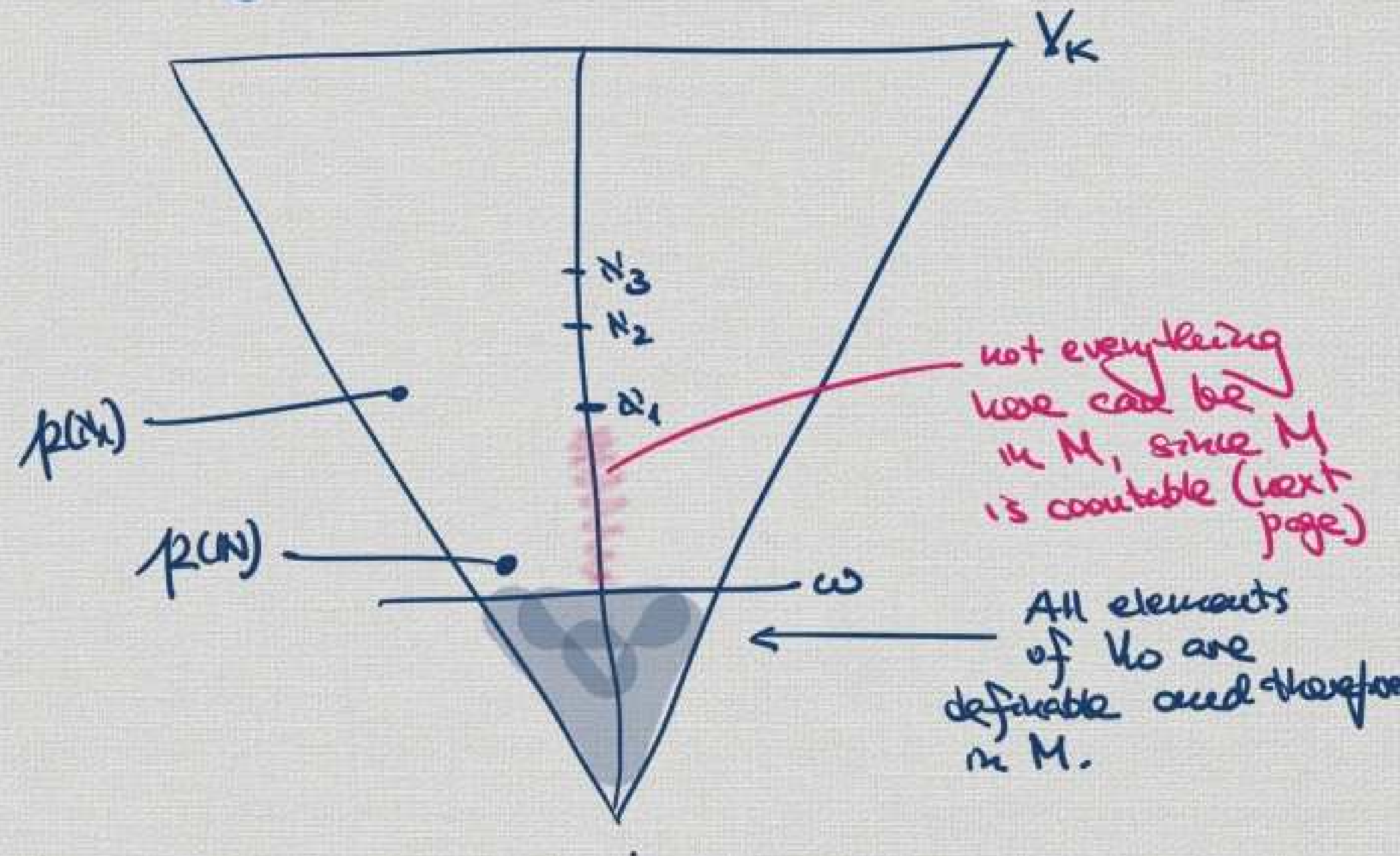
since there are finitely many, there is some $j \in \mathbb{N}$ s.t.

$$\vec{p} \in M_j^{<\omega}$$

Now $V_\kappa \models \exists x \varphi(x, \vec{p})$ for $\vec{p} \in M_j^{<\omega}$

Therefore $w(\varphi, \vec{p}) \in M_{j+1} \subseteq M$.

So by TV, $M \preccurlyeq V_\kappa$. q.e.d.



Note that by induction

$M_i$ is countable

and thus $M = \bigcup_{i \in \omega} M_i$ is also countable.

[The step from $M_i$ to $M_{i+1}$ adds at most $\underbrace{\aleph_0 \cdot |M_i^{<\omega}|}_{=\aleph_0}$ many elts.]

Therefore $M \preccurlyeq V_\kappa$ and countable, but certainly not transitive since $\aleph_1 \in M$, but $\aleph_1 \not\subseteq M$.

How can we make a non-transitive set transitive.

Mostowski's Collapsing Theorem

If $M \subseteq V_\kappa$ s.t. $M \models \mathrm{Ext}$, then there is a transitive set $T$ s.t. $(T, \in) \cong (M, \in)$.

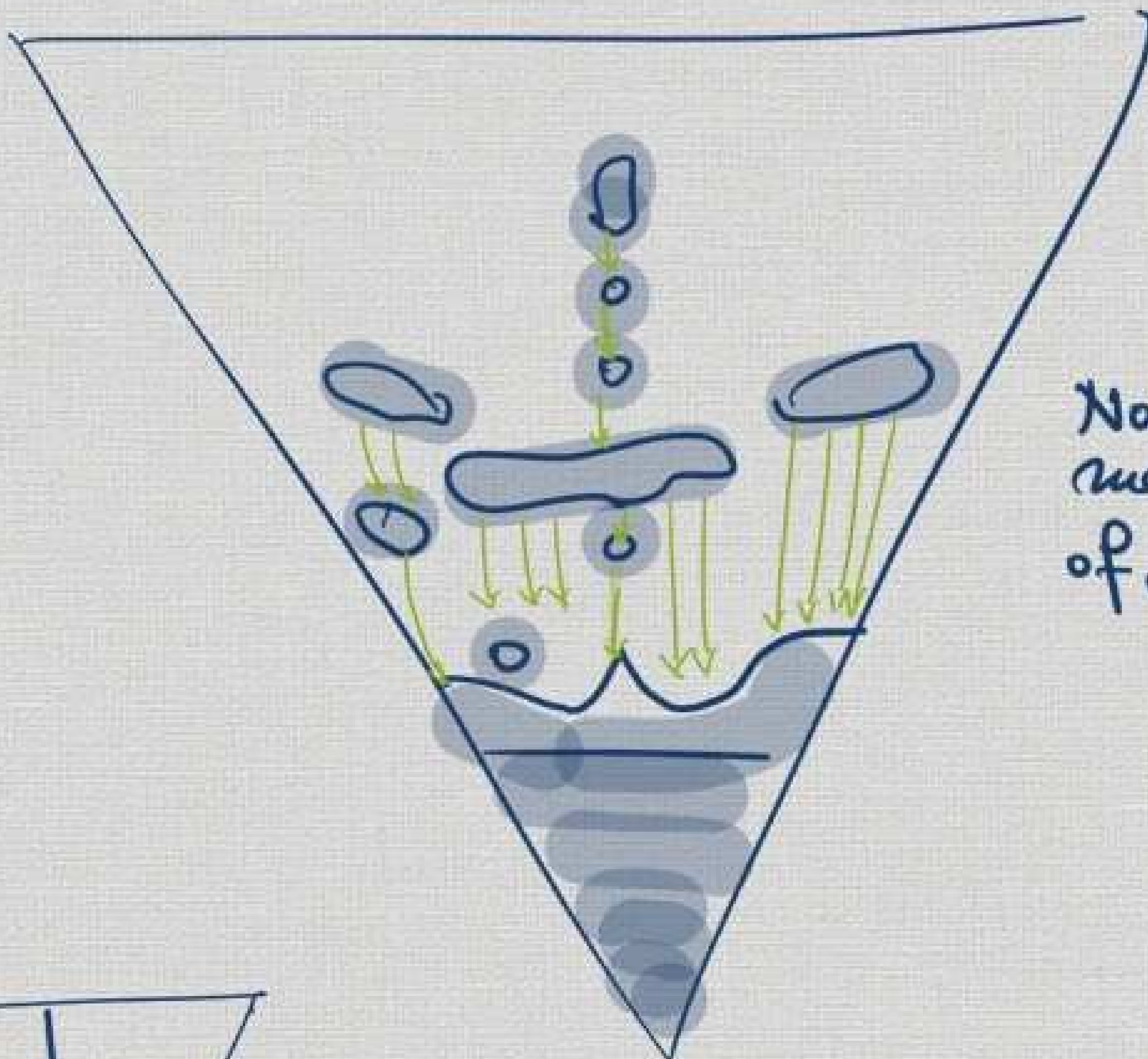
[Proof. By $\in$-recursion.

$$f(x) := \{ f(y) ; y \in x \}.$$

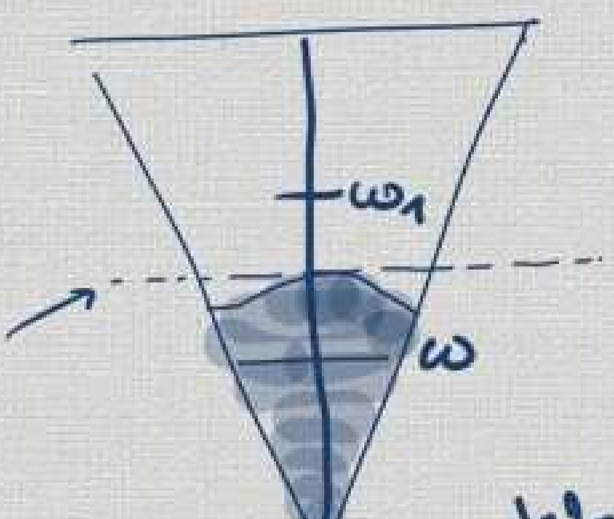
Then $T := \mathrm{im}(f)$ is transitive and

$$f : (M, \in) \cong (T, \in).$$

Here we use $M \models \mathrm{Ext}$ for injectivity of the function.]



Non-transitive model with lots of gaps.




We could call $\mathrm{Ord} \cap T$ the <u>height of the model</u>

We obtain that

$$f : M \longrightarrow T$$
$$id : M \longrightarrow V_\kappa$$

are both elementary

Thus $f^{-1} : T \longrightarrow V_\kappa$ is an elementary embedding.

In particular: $(T, \in) \models ZF$.

But $T \cong M$, so $|T| = |M|$, so $T$ is countable, and therefore the height of $T$ is a countable ordinal.

In particular,

$T \models$ there are uncountable cardinals
in particular $\aleph_1, \aleph_2, \aleph_3, \aleph_4$ etc.

All of these are countable ordinals.

<u>Corollary</u> The formula $K(x) :=$ "$x$ is a cardinal" is not absolute between $T$ and $V_\kappa$.

<u>Cardinals</u>

$0, 1, 2, 3, 4, \ldots$
$\aleph_0$
$\aleph_1, \aleph_2, \aleph_3, \aleph_4 \ldots$

absolute for arbitrary trs model

finite or countable

uncountable cardinals

<u>Remark</u>

While this shows that a transitive inner model and the outer model can disagree about cardinals, <u>this</u> technique always produces

$T \equiv V_\kappa$ [i.e., for all sentences $\sigma$
$T \models \sigma \iff V_\kappa \models \sigma$ ],

so in particular
$V_\kappa \models CH \iff T \models CH.$

**Theorem** Suppose $M$ is transitive in $N$.

(i) If $\mathcal{P}(\omega)^M = \mathcal{P}(\omega)^N$, then $\aleph_1^M = \aleph_1^N$.

This notation means:
the formula defining $\mathcal{P}(\omega)$

$\Phi$ [ i.e., the unique object that consists of all subset of the unique object that is the smallest inductive set

relativised to $M$ or $N$, resp., i.e.
$\Phi^M$ or $\Phi^N$.

(ii) If $\mathcal{P}(\omega)^M = \mathcal{P}(\omega)^N$, then $N \models \neg CH$
$\Longrightarrow M \models \neg CH$.

**Proof.** (i). In Lecture I we proved that there is a surjection from $\mathcal{P}(\omega)$ onto $\aleph_1$ by producing $R \subseteq \omega \times \omega$ s.t. $(\omega, R)$ is wellordered.

If $M$ is transitive in $N$, since being a cardinal is downwards absolute, know if $\aleph_1^M \neq \aleph_1^N$, then $\aleph_1^M < \aleph_1^N$. So $\aleph_1^M$ is a countable ordinal in $N$, so there is $R \in N$ s.t.

$$N \models (\omega, R) \cong (\aleph_1^M, \in).$$

By assumption, $R \in M$ so by absoluteness of wellfoundedness

$M \models (\omega, R)$ is wellfounded, so

$M \models (\omega, R) \cong (\alpha, \in)$ for some $\alpha$,

so $\alpha = \aleph_1^M$ and thus $M \models \aleph_1^M$ is countable.

Contradiction!

(ii) Suppose $M \models CH$.

i.e., $M \models \exists f : \mathcal{P}(\omega) \longrightarrow \aleph_1$ bijection

By assumption $\mathcal{P}(\omega)^M = \mathcal{P}(\omega)^N$.

By (i), $\aleph_1^M = \aleph_1^N$.

Since $M \subseteq N$, $f \in N$.

So $N \models \exists f : \mathcal{P}(\omega) \longrightarrow \aleph_1$ bijection.

Thus $N \models CH$.

q.e.d.

Remark So, if I want to make CH true in an inner model, I cannot include all subsets of $\omega$. [Unless we already had CH in the bigger model.]

Theorem If $M \subseteq N$ is transitive s.t. "$x$ is a cardinal" is absolute between $M$ and $N$.

Then $N \models CH \implies M \models CH$.

Proof. Suppose $M \models \neg CH$, i.e., there is some surjection $f : \mathcal{P}(\omega) \longrightarrow \aleph_2^M$ in $M$.

But by assumption, $\aleph_1^M = \aleph_1^N$ and $\aleph_2^M = \aleph_2^N$ and $f \in N \supseteq M$, so $N \models f : \mathcal{P}(\omega) \longrightarrow \aleph_2$ is a surj.

$\implies N \models \neg CH$.

q.e.d.

<u>Corollary</u> If $M \subseteq N$ and

$\mathcal{P}(\omega)^M = \mathcal{P}(\omega)^N$ and

"x is a cardinal" is absolute between $M$ & $N$

then $M \models CH \iff N \models CH$.