The Constructible Universe III

THIRD LECTURE
22 April 2024

Recap ① Very few formulas absolute for substructures in the language of set theory

② Δ_0-formulas absolute for TRANSITIVE substructures

③ Absoluteness closed under concatenation etc

Absolute $\emptyset$, union, intersection, function, surjection, injection, bijection

Not yet discussed Ordinal, cardinal.

Note that
$V = \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha$
is a definable inner model, definable by formula
$\exists \alpha \, (\alpha \text{ is an ordinal} \wedge \underline{x \in V_\alpha})$

That this is a formula is the proof of the recursion theorem.

§6 von Neumann hierarchy - CUMULATIVE HIERARCHY

$V_0 := \emptyset$

$V_{\alpha+1} := \mathcal{P}(V_\alpha)$

$V_\lambda := \bigcup_{\alpha<\lambda} V_\alpha$ $\quad \lambda$ limit ordinal

We know: ① All V_α are transitive.

② $\rho(x) := \min\{\alpha ;\ x \in V_{\alpha+1}\}$

$\rho(x) = \sup\{\rho(y)+1 ;\ y \in x\}$

Remark This is usually used to prove $\mathrm{Cons}(ZF^-) \longrightarrow \mathrm{Cons}(ZF)$

ZF^-: ZF without Foundation

This is precisely an inner model argument as sketched on the last page.

Q If α is any ordinal which axioms of set theory hold in $(V_\alpha, \in)$?

This is again an "inner model argument" which we will see in Lecture III.

Let's see the proof of $V \models ZF$.

ZF^- : Zermelo-Fraenkel w/o Foundation
ZF : Zermelo-Fraenkel w/ Foundation

We're going to prove:
in each model $M \models ZF^-$, the formula
$\Phi(x) :\Leftrightarrow x \in V$, $\Leftrightarrow \exists \alpha\ x \in V_\alpha$
$\Leftrightarrow \exists \alpha$ there is a fn f with $\mathrm{dom}(f) \in \mathrm{Ord}$ and $\alpha \in \mathrm{dom}(f)$ and f satisfies the Rec. eq. for V below $\alpha + 1$ and $x \in f(\alpha)$.

defines an inner model of ZF

Axioms of ZF:
Pairing, Union, Powerset
Ext, Separation, Replacement,
extra $\to$ Foundation, Infinity

Most of these are functional: $\forall x \exists Z$ s.t. ...

For these, we only need to show that if $x, y \in V$ then so is
$\{x, y\} \in V$
$\bigcup x \in V$
$\wp(x) \in V$.

Reminder
$\alpha \leq \beta \Rightarrow V_\alpha \subseteq V_\beta$
V_α is transitive
$(x \in V_\alpha \Rightarrow x \subseteq V_\alpha)$

In particular, if $x, y \in V_\alpha$, then $\{x, y\} \subseteq V_\alpha$

$\Longrightarrow \{x, y\} \in V_{\alpha+1}$.

Reminder:

$V_\alpha \cap \mathrm{Ord} = \alpha$

$\Longrightarrow \omega \in V_{\omega+1}$

(*)

$x \in V_\alpha \overset{\text{trs}}{\Longrightarrow} x \subseteq V_\alpha$

$\Longrightarrow \bigcup x \in V_{\alpha+1}$.

$x \in V_\alpha \Longrightarrow$ if $y \subseteq x$, then $y \subseteq V_\alpha$, so $y \in V_{\alpha+1}$.

Therefore $\wp(x) \in V_{\alpha+2}$

So $V \subseteq M$ is a model of

1. Extensionality (last time)
2. Pairing, Union, Powerset (above)
3. Infinity (by (*))
4. Separation

If $x \in V$, then by Separation in M, I can form

$$\{y \in x ;\ M \models \varphi(y, \vec{p})\} \in V$$

[since $\wp(x) \subseteq V$]

Remark: $\{y \in x ;\ V \models \varphi(y, \vec{p})\} \in V_{\alpha+1}$

But that's not $V \models$ Separation.

For this, I need $\{y \in x ;\ V \models \varphi(y, \vec{p})\}$

Relativisation: $\{y \in x ;\ V \models \varphi(y, \vec{p})\} = \{y \in x ;\ M \models \varphi^V(y, \vec{p})\} \in V$.

(5) Replacement
similar to Separation

(6) Foundation.

[Remark: in Lecture II we proved that transitive submodels of models of Found satisfy Found. That doesn't help here!]

Need to show

$V \models \forall x \exists m \, (m \in x \wedge m \cap x = \emptyset)$

Fix $x \in V$. So find α s.t. $x \in V_\alpha$. So $\rho(x) < \alpha$.

and $\{\rho(y) ; y \in x\} \subseteq \rho(x) < \alpha$.

Since this is a set of ordinals, find $\alpha_0 \in \{\rho(y) ; y \in x\}$ minimal. So there is some y_0 s.t. $\rho(y_0) = \alpha_0$.

If $z \in y_0 \cap x$, then $\rho(z) < \alpha_0$,
and so $z \notin x$.

q.e.d.

Corollary (to the proof).

If λ is a limit ordinal, then Pair, Powerset, Union, Separation are all true in V_λ.

For any α, $V_\alpha \models$ Extensionality and Foundation.

For any $\alpha \geq \omega+1$, $V_\alpha \models$ Infinity.

What about Replacement?

That's not true, e.g., in $V_{\omega+\omega}$.

Replacement says: if $f: V_{\omega+\omega} \longrightarrow V_{\omega+\omega}$ is definable and $x \in V_{\omega+\omega}$, then

$$\{f(y) ; y \in x\} \in V_{\omega+\omega}.$$

Counterexample 1. $f(z) := \begin{cases} \omega+z & \text{if } z \in \omega \\ 0 & \text{o/w} \end{cases}$

Clearly definable.

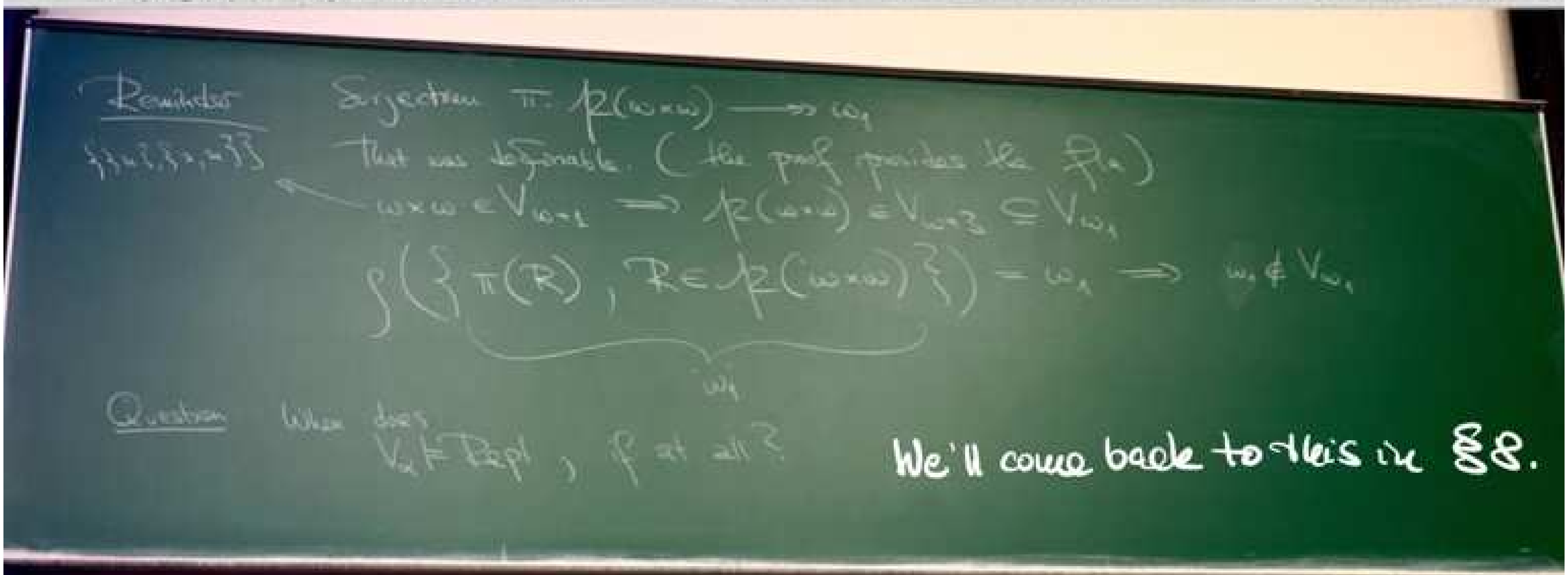
$\omega \in V_{\omega+\omega}$; $\{f(y) ; y \in \omega\} =$

$\{\omega+n ; n \in \omega\} = R$

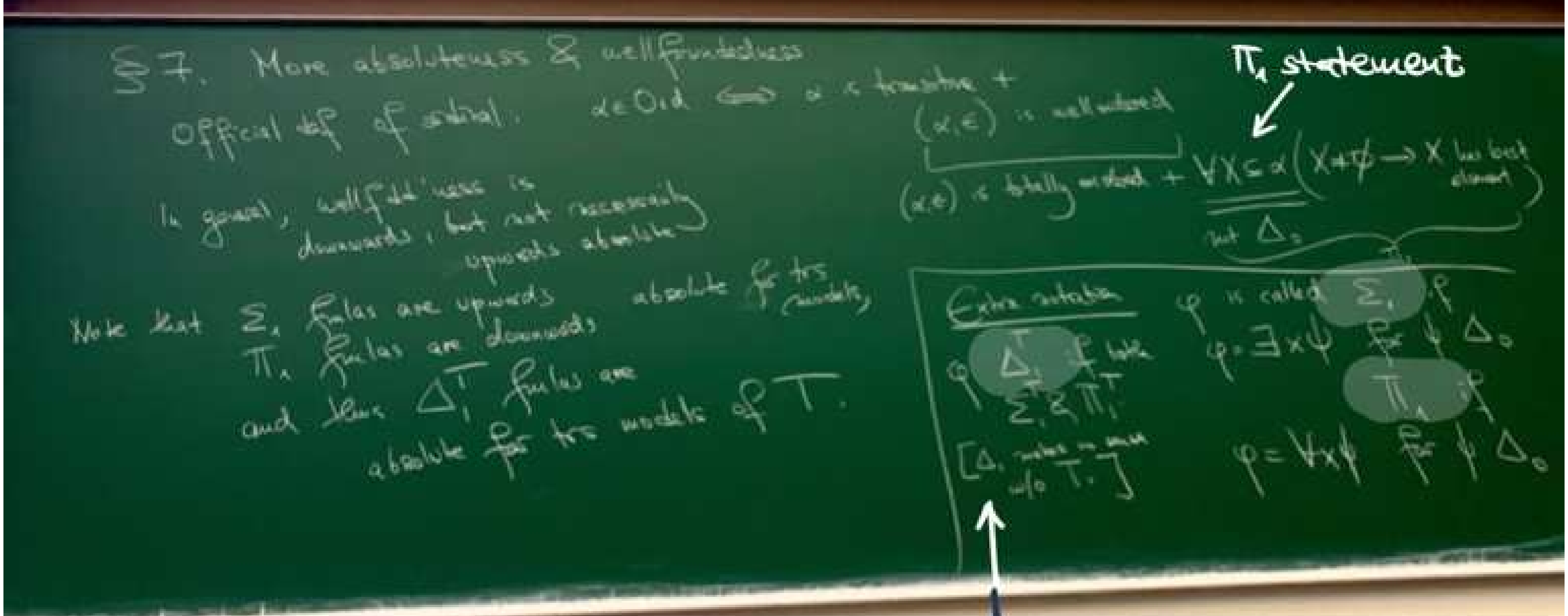
$\rho(R) = \omega+\omega \notin V_{\omega+\omega}$.

Counterexample 2. $V_{\omega_1} \not\models$ Replacement.

Reminder: Lecture I



We'll come back to this in §8.



Π_1 statement

No formula can be both Σ_1 & Π_1, but they can be Σ_1^T & Π_1^T.

The AV system in H1 broke down in the middle of the lecture and we had to switch to the blackboard.

A side remark

Wellfoundedness of a structure is not expressible in first-order logic

Let $(M, E) \models ZF$. Add ctbly many constants $c, c_0, c_1, c_2, \ldots$ ("c is the least non-zero limit ordinal")

$\Phi_n = ZF \cup \{\varphi_i ; i < n\}$

φ_0: c

φ_1: $c_0 \in c$

φ_2: $c_1 \in c_0 \in c$

φ_3: $c_2 \in c_1 \in c_0 \in c$

$\vdots$

Φ_n is satisfiable.

Take structure M with c interpreted as ω, c_i as $\omega - i$ for $i < n$.

So, by compactness $\bigcup_{n \in \omega} \Phi_n$ is satisfiable.

But there is an illfounded model of ZF (worse, it has a decreasing sequence [illegible])

Key observation $M \models$ Foundation does not imply that M itself is wellfounded

This is because wellfoundedness of ANY set follows from Foundation.

However, in ZF, we have

α is ordinal $\iff$ α is transitive & $(\alpha, \in)$ is totally ordered

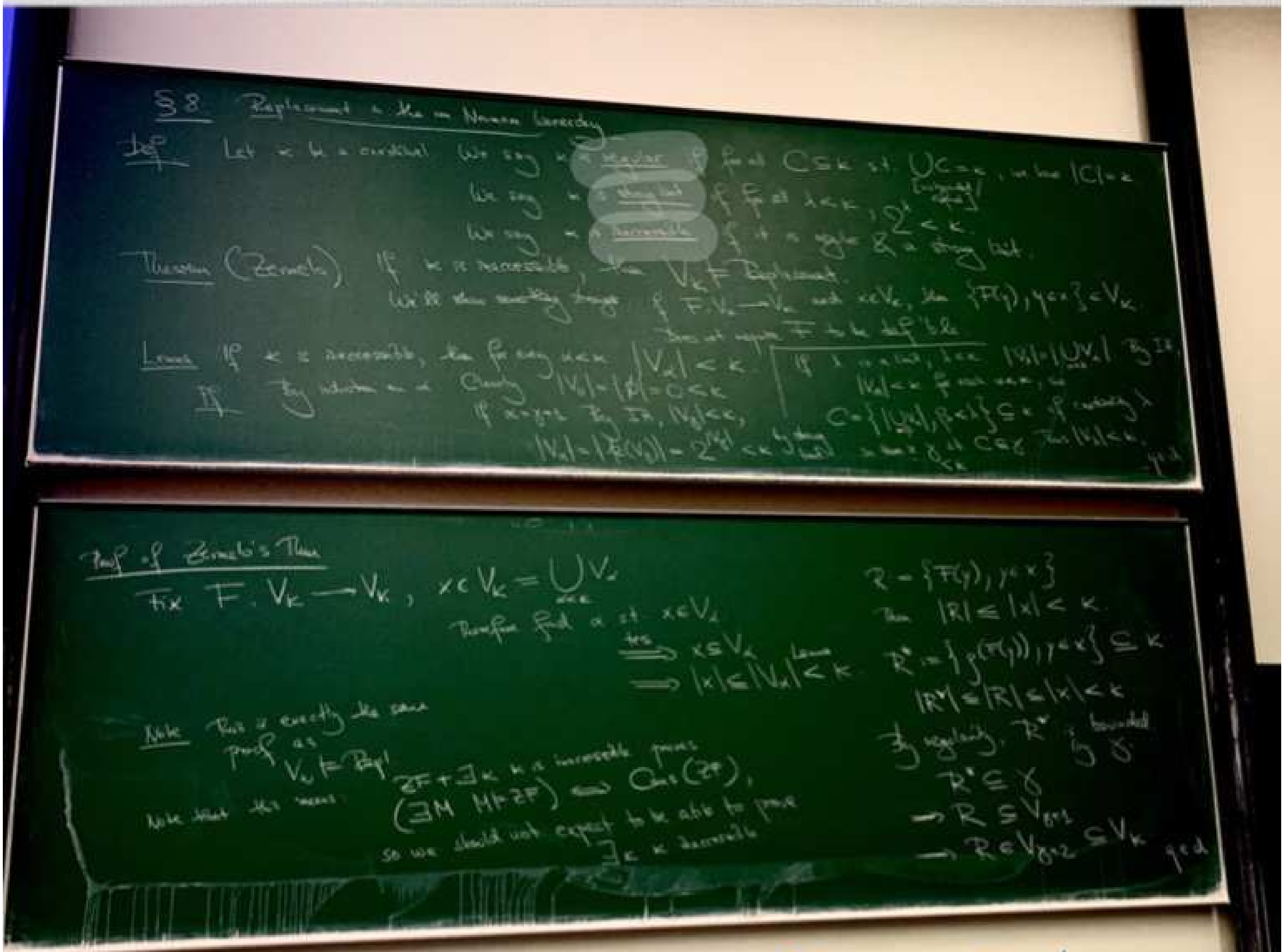
Therefore, if $M \subseteq N$ is transitive and both satisfy Foundation, then "α is an ordinal" is absolute between M & N.

What about "α is a cardinal"? $\underbrace{\forall \beta < \alpha}_{\text{bounded quantifier}} \underbrace{\forall f : \beta \to \alpha}_{\text{is not obviously bounded}}$ f is not a bijection

Note. This does not prove yet, that "α is a cardinal" is not absolute. We'll have to do this (later).

We'll prove this in §9.

NOTE The next lecture will be ONLINE! Zoom link is sent by e-mail.



κ is **INACCESSIBLE** if it's a regular strong limit

Lemma If κ is inaccessible, then $\forall \alpha < \kappa$ $|V_\alpha| < \kappa$.

Theorem If κ is inaccessible, then

$V_\kappa \models$ Replacement.

Reminder: Surjection $\pi : \mathcal{P}(\omega\times\omega) \twoheadrightarrow \omega_1$

That was definable. (the proof provides the fla)

$\{\{n\},\{n,m\}\} \leftarrow \omega\times\omega \in V_{\omega+1} \implies \mathcal{P}(\omega\times\omega) \in V_{\omega+3} \subseteq V_{\omega_1}$.

$$\rho\Big(\underbrace{\{\pi(R),\ R\in\mathcal{P}(\omega\times\omega)\}}_{\omega_1}\Big) = \omega_1 \implies \omega_1 \notin V_{\omega_1}$$

Question: When does $V_\alpha \models \mathrm{Repl}$, if at all?

§7. More absoluteness & wellfoundedness

Official def of ordinal: $\alpha\in\mathrm{Ord} \iff \alpha$ is transitive + $\underbrace{(\alpha,\in) \text{ is wellordered}}$

$(\alpha,\in)$ is totally ordered + $\underbrace{\underline{\underline{\forall X\subseteq\alpha}}\,\big(X\neq\emptyset \implies X \text{ has least element}\big)}_{\Pi_1}$

not Δ_0

In general, wellfdd'ness is downwards, but not necessarily upwards absolute.

Note that Σ_1 fmlas are upwards absolute for trs models, Π_1 fmlas are downwards, and thus Δ_1^T fmlas are absolute for trs models of T.

Extra notation

φ is called Σ_1 if $\varphi = \exists x\psi$ for ψ Δ_0

Π_1 if $\varphi = \forall x\psi$ for ψ Δ_0

φ Δ_1^T if both Σ_1^T & Π_1^T.

[Δ_1 makes no sense w/o T.]

A side remark

Wellfdd'ness of a structure is not expressible in first-order logic

Let $(M, E) \models ZF$. Add ctbly many constants $c, c_0, c_1, c_2, \ldots$

φ_0: $c = \omega$ ("c is the least non-zero limit ordinal")

φ_1: $c_0 \in c$

φ_2: $c_1 \in c_0 \in c$

φ_3: $c_2 \in c_1 \in c_0 \in c$

$\vdots$

$\Phi_n := ZF \cup \{\varphi_i ; i < n\}$

Φ_n is satisfiable: Take structure M with c interpreted as ω, c_i as $n-i$ for $i \leq n$.

So, by compactness $\bigcup_{n \in \omega} \Phi_n$ is satisfiable.

But this is an illfounded model of ZF (worse: it has a decreasing sequence inside its ω).

<u>Key observation</u> $M \models$ Foundation does not imply that M itself is wellfounded.

However, in ZF, we have

α is ordinal $\iff$ α is transitive & $(\alpha, \in)$ is totally ordered

Therefore, if $M \subseteq N$ is transitive and both satisfy Foundation, then "α is an ordinal" is absolute between M & N.

What about "α is a cardinal"? $\forall \beta < \alpha \ \forall f : \beta \longrightarrow \alpha$ f is not a bijection

bounded quantifier

is not obviously bounded

<u>Note</u>: This does not prove yet, that "α is a cardinal" is not absolute. We'll have to do this (later).

§8. Replacement in the von Neumann hierarchy

Def. Let κ be a cardinal. We say κ is <u>regular</u> if for all $C \subseteq \kappa$ st. $\bigcup C = \kappa$ [unbounded / cofinal], we have $|C| = \kappa$.

We say κ is <u>strong limit</u> if for all $\lambda < \kappa$, $2^\lambda < \kappa$.

We say κ is <u>inaccessible</u> if it is regular & a strong limit.

Theorem (Zermelo). If κ is inaccessible, then $V_\kappa \models$ Replacement.

We'll show something stronger: if $F: V_\kappa \to V_\kappa$ and $x \in V_\kappa$, then $\{F(y), y \in x\} \in V_\kappa$.

Does not require F to be def'ble.

Lemma If κ is inaccessible, then for every $\alpha < \kappa$ $|V_\alpha| < \kappa$.

Pf By induction on α. Clearly $|V_0| = |\emptyset| = 0 < \kappa$.

If $\alpha = \gamma + 1$ By IH, $|V_\gamma| < \kappa$,

$|V_\alpha| = |\mathcal{P}(V_\gamma)| = 2^{|V_\gamma|} < \kappa$ by strong limit.

If λ is a limit, $\lambda < \kappa$. $|V_\lambda| = |\bigcup_{\alpha<\lambda} V_\alpha|$. By IH, $|V_\alpha| < \kappa$ for each $\alpha < \kappa$, so

$C = \{ |\bigcup_{\alpha<\beta} V_\alpha|, \beta < \lambda \} \subseteq \kappa$ of cardinality λ

so there is $\underset{<\kappa}{\gamma}$ s.t. $C \subseteq \gamma$. Thus $|V_\lambda| < \kappa$.

qed

Proof of Zermelo's Thm

Fix $F: V_\kappa \to V_\kappa$, $x \in V_\kappa = \bigcup_{\alpha<\kappa} V_\alpha$

Therefore find α st. $x \in V_\alpha$.

$\overset{\text{trs}}{\Longrightarrow} x \subseteq V_\alpha$

$\Longrightarrow |x| \leq |V_\alpha| \overset{\text{Lemma}}{<} \kappa$.

$R := \{F(y), y \in x\}$

Then $|R| \leq |x| < \kappa$.

$R^* := \{\rho(F(y)), y \in x\} \subseteq \kappa$

$|R^*| \leq |R| \leq |x| < \kappa$

By regularity, R^* is bounded by γ.

$R^* \subseteq \gamma$

$\Longrightarrow R \subseteq V_{\gamma+1}$

$\Longrightarrow R \in V_{\gamma+2} \subseteq V_\kappa$ qed

Note This is exactly the same proof as $V_\omega \models$ Repl.

Note that this means: $ZF + \exists \kappa$ κ is inaccessible proves $(\exists M \; M \models ZF) \iff \mathrm{Cons}(ZF)$,

so we should not expect to be able to prove $\exists \kappa$ κ inaccessible.