The Constructible Universe

Lecture II

15 April 2024

<u>Recap</u> Technique of inner models
Start with $V \models ZF$ and form inner model
$M \subseteq V$ s.t. $M \models ZFC + CH$.

Substructures in model theory: $M \subseteq N$ substructure
Propositional fmlas absolute between M, N
$\exists$-fmlas upwards abs.
$\forall$-fmlas downwards abs.

<u>Problem</u> In $\mathcal{L}_\in$ there are no interesting propositional fmlas.
So no reason to assume that $M \subseteq V$ satisfies <u>ANY</u>thing.
Even "being $\emptyset$" is not preserved.

§4 Transitive substructures

From: Lecture I / Last page.

Def. If (M,E) is an $\mathcal{L}_\in$-structure, we say that $N \subseteq M$ is a transitive substructure if for all $x \in N$ & $y \in M$, if yEx, then $y \in N$.

[That's just a model-theoretic reformulation of the standard notion of "transitive set".].

If $N \subseteq M$ is transitive, then Φ_0 is absolute between N & M.

Proof. Since Φ_0 was $\forall$-formula, it is downwards absolute. Thus only need to show upwards.

Suppose not:
$(N,E) \models \Phi_0(x)$ & $\underbrace{(M,E) \models \neg\Phi_0(x)}$

$\Downarrow$

there is $y \in M$ s.t. yEx

$y \in N$ (by transitivity)

$\Longrightarrow (N,E) \models \neg\Phi_0(x)$.

Contradiction! q.e.d.

Extend absoluteness for transitive substructures:

<u>Def.</u> Define the class of Δ_0-formulas by recursion as the smallest class containing all atomic fmlas ($x \in y$, $x = y$), closed under propositional connectives, and closed under:

if φ is Δ_0, then so is

$$\underbrace{\exists x (x \in y}_{\text{BOUNDED QUANTIFICATION}} \wedge \varphi)$$

Abbreviated as $\exists x \in y \, \varphi$.

<u>Theorem</u> If M is transitive in N, then all Δ_0-formulas are absolute between M, N.

<u>Proof.</u> By induction on rec. def. of Δ_0.

1. Atomic formulas abs. for all substructures.
2. Proved last time when we did this for arbitrary substructures.
3. Clearly if φ is absolute, then $\exists x (x \in y \wedge \varphi)$ is [as an $\exists$-formula] upwards absolute.

Suppose $N \models \exists x (x \in a \wedge \varphi)$ for some $a \in M$

Therefore there is $\boxed{b \in N}$ s.t. $N \models \boxed{b \in a} \wedge \varphi$

$b \in a$

trs $\Rightarrow b \in M \Longrightarrow M \models \exists x (x \in a \wedge \varphi)$ qed

We saw this in action in our example:

Empty set formula:

$$
\begin{aligned}
x=\emptyset &\iff \forall z\,(z\notin x)\\
&\iff \neg\exists z\,\neg\,(z\notin x)\\
&\iff \neg\exists z\ \underbrace{z\in x}\\
&\iff \neg\exists z\,(z\in x\wedge z=z)
\end{aligned}
$$

So, this is a negation of a boundedly quantified formula, therefore a Δ_0-formula.

Let T be any theory, we say that φ is Δ_0^T if there is a Δ_0-formula ψ s.t.

$$T\vdash \varphi\leftrightarrow\psi.$$

Then we have that if $M\subseteq N$, M is transitive in N and $M,N\models T$, then Δ_0^T-formulas are absolute between M,N.

This gives us the following alternative bounded quantification:

abbreviated as $\forall x\in y\ \varphi$

$$
\begin{aligned}
&\forall x\,(x\in y\rightarrow\varphi)\\
\iff& \neg\exists x\,\neg\,(x\in y\rightarrow\varphi)\\
\iff& \neg\exists x\,\neg\,(x\notin y\vee\varphi)\iff\neg\exists\,(x\in y\wedge\neg\varphi)
\end{aligned}
$$

get $\exists x \in y\, \varphi \iff \neg \forall x \in y\, \neg \varphi$.

A list of Δ_0^T-formulas for $T =$ Predicate Logic.

1. $x \in y$
2. $x = y$
3. $x \subseteq y \iff \forall z \in x\, (z \in y)$
4. $z = \{x, y\} \iff x \in z \wedge y \in z \wedge \forall w \in z\, (w = x \vee w = y)$
5. $z = \{x\}$
6. $z = (x, y) = \{\{x\}, \{x, y\}\}$
7. $z = \emptyset$
8. $z = x \cup y \iff x \subseteq z \wedge x \subseteq y \wedge \forall w \in z\, (w \in x \vee w \in y)$
9. $z = x \cap y$
10. $z = x \setminus y$
11. $z = x \cup \{x\}$
12. $\varphi(x) =$ x is transitive
13. $z = \bigcup x$

Important to note

This only means that the formulas describing these are absolute, not the corresponding existence axioms

$\forall x \forall y \exists p\, (p = \{x, y\})$

E.g.) $\{\emptyset\}$ is transitive, but

$$\forall x \exists s \,(s = \{x\})$$

is false in $\{\emptyset\}$, but can be true in N.

Let's start our analysis of transitive substructures with the "structural" axioms:

Extensionality $\forall x \forall y \,[x = y \leftrightarrow \forall w\,(w \in x \leftrightarrow w \in y)]$

Foundation.

(1) Extensionality $\forall x \forall y \; x = y \Longleftrightarrow (x \subseteq y \wedge y \subseteq x)$

Suppose $(N, E) \models \mathrm{Ext}$. Can we show $(M, E) \models \mathrm{Ext}$ if $M \subseteq N$ is transitive?

Clearly, since Δ_0 is absolute and the axiom is of the form $\forall \psi$ with ψ Δ_0, we get downwards absoluteness.

(2) Foundation $\forall x \exists m \,(m \in x \wedge \forall w \,\neg (w \in m \wedge w \in x))$

Suppose $(N, E) \models$ Foundation and

$(M, E) \models \neg$ Foundation

with M transitive.

So, there $a \in M$ without minimal element in M.

Clearly $a \in N \supseteq M$ and by Foundation in N, we have a $\underline{b \in N}$ that is minimal in N:

$$N \models \underbrace{\forall z \neg (z \in b \wedge z \in a)}$$

$\underline{\text{st. } bEa}$ once more $\forall$ applied to Δ_0, so downwards absolute

$\xRightarrow{\text{ts}} b \in M$

$$\Longrightarrow M \models \forall z \neg (z \in b \wedge z \in a)$$

Thus b is E-minimal in M, and so we obtain a contradiction!

$\forall z\,(z \notin b \vee z \notin a)$
$\forall z\,(z \in b \rightarrow z \notin a)$
$\forall z \in b\,(z \notin a)$
$\neg \exists z \in b\,(z \in a)$

<u>Summary</u> If we have (N, E) a model of Ext + Foundation, then (M, E) where M is ts in N will also satisfy Ext + Foundation.

<u>Remark</u> The same argument gives us preservation of wellfoundedness between ts submodels and the big model.

<u>Observation</u> If $M \subseteq N$ is transitive and N is a model of pairing axiom and for each $x, y \in M$, $\{x, y\} \in M$ then $M \models$ Pairing Axiom.

Similarly for the Union Axiom.

This is an equivalence:

if for some $x, y \in M$, $\{x, y\} \notin M$, then $M \not\models$ Pairing & similarly for Union, since

$$z = \{x, y\} \text{ is } \Delta_0.$$

This is slightly different for power set:

if for $x \in M$, $P(x) \in M$, then $M \models$ Power set,

but this is not necessary as our example of

$$(\mathbb{N}, \in) \models \text{Power set}$$

from Lecture I shows.

An operation F on a model N is called definable if there is a formula Φ s.t.

$$z = F(x_1, \dots, x_n) \iff N \models \Phi(z, x_1, \dots, x_n)$$

We say that such an operation F is absolute between M & N if the formula Φ is absolute.

That means that $F(x,y) := \{x,y\}$ is an operation under the assumption of the pairing axiom & then absolute by the previous observation.

<u>Lemma</u> If φ is absolute and $F, G_1, \dots, G_n$ are absolute operations, then

$$\psi(x_1, \dots, x_m) :\iff \varphi(G_1(x_1, \dots, x_m), \dots, G_n(x_1, \dots, x_m))$$

$$H(x_1, \dots, x_m) := F(G_1(x_1, \dots, x_m), \dots, G_n(x_1, \dots, x_m))$$

are absolute.

<u>Proof.</u> Just check the definitions. $\triangleleft$

Remark. Concatenation of Δ_0 formulas and operations still gives absolute formula & operations but not necessarily Δ_0.

Kunen's book

$F(n) = n+n$ for $n \in \mathbb{N}$

$F(x) = 0$ for $x \notin \mathbb{N}$

can be written as concatenation of Δ_0 but not as Δ_0.

Further absolute notions:

$\bigcup \{\{x\},\{x,y\}\} = \{x,y\} = \{x\} \cup \{x,y\}$

1. z is an ordered pair
2. $z = a \times b$
3. z is a relation
4. $\mathrm{dom}(z)$
5. $\mathrm{ran}(z)$
6. z is a function
7. z is an injection
8. z is a surjection
9. z is a bijection

§5 Relativisation

<u>Def.</u> $M \subseteq N$ is a definable substructure if there is a formula Φ s.t.
$$x \in M \iff N \models \Phi(x).$$

For definable substructures, we can talk about truth in M from the point of view of N:

Define φ^{Φ} by recursion:

$$(x=y)^{\Phi} := x=y$$
$$(x\in y)^{\Phi} := x\in y$$
$$(\varphi\wedge\psi)^{\Phi} := \varphi^{\Phi}\wedge\psi^{\Phi}$$
$$(\neg\varphi)^{\Phi} := \neg\varphi^{\Phi}$$
$$(\exists x\varphi)^{\Phi} := \underbrace{\exists x\,(\Phi(x)}_{\text{Relativisation of the quantifier}}\wedge\varphi^{\Phi})$$

By simple induction, get
$$(M,E)\models\varphi \iff (N,E)\models\varphi^{\Phi}.$$

This makes the annoying talk of logicians (e.g., "N knows that φ is true in M") much less mysterious.

So, e.g., if M is transitive, then

$$N \models \mathrm{Ext}^M$$

[If M is definable by fmla Φ, we also write φ^M for φ^Φ.]

<u>Corollary</u> Suppose T, S are two theories s.t. there is a formula Φ s.t.

$$T \vdash \varphi^\Phi \text{ for any } \varphi \in S.$$

Then $\mathrm{Cons}(T) \Longrightarrow \mathrm{Cons}(S)$.

<u>Proof.</u> Suppose $\neg\mathrm{Cons}(S)$, so

$$S \vdash \varphi \wedge \neg\varphi.$$

Assuming $\mathrm{Cons}(T)$, we have $N \models T$ $\Longrightarrow$ $N \models (\varphi \wedge \neg\varphi)^\Phi$

$\Longrightarrow N \models \varphi^\Phi$ and $N \models (\neg\varphi)^\Phi$

[$\Longleftrightarrow N \models \neg\varphi^\Phi$]

Together $N \models \varphi^\Phi \wedge \neg\varphi^\Phi$

So, T is inconsistent.

q.e.d.

§6 von Neumann hierarchy – CUMULATIVE HIERARCHY

$$V_0 := \emptyset$$
$$V_{\alpha+1} := \mathcal{P}(V_\alpha)$$
$$V_\lambda := \bigcup_{\alpha<\lambda} V_\alpha \quad \lambda \text{ limit ordinal}$$

We know: ① All V_α are transitive.

② $\rho(x) := \min\{\alpha ; x \in V_{\alpha+1}\}$

$\rho(x) = \sup\{\rho(y)+1 ; y \in x\}$

Remark This is usually used to prove $\mathrm{Cons}(ZF^-) \Longrightarrow \mathrm{Cons}(ZF)$

ZF^-: ZF without Foundation

This is precisely an inner model argument as sketched on the last page.

Q If α is any ordinal which axioms of set theory hold in $(V_\alpha, \in)$?

This is again an "inner model argument" which we will see in Lecture III.

From Lecture II

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We know:

1. All V_α are transitive.
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