The Constructible Universe

Lecture I

8 April 2024

Lectures I – III (8 – 22 April) **IN PERSON**
Lectures IV – X (29 April – 24 June) **ONLINE**
Lectures XI & XII (1 & 8 July) **IN PERSON**

Exams: Oral Exam
probably late July 2024

GOAL Prove that AC and CH are consistent (relative to ZF).

Axiom of Choice

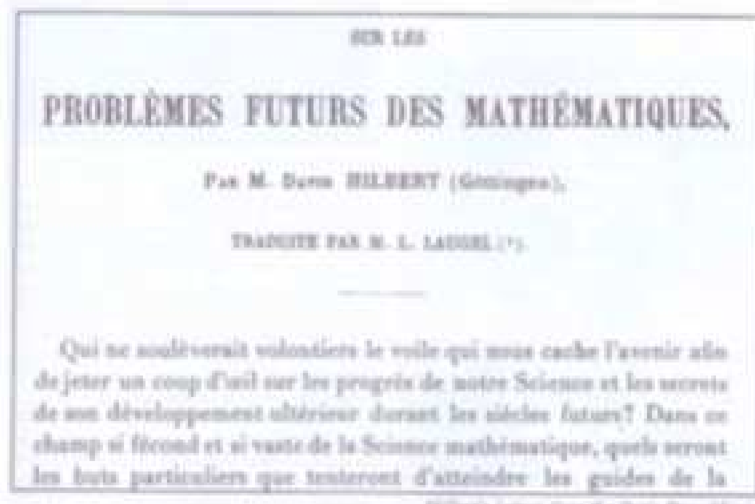
Continuum Hypothesis

§1 Historical background & motivation

David Hilbert (1862-1943)

Hilbert's Problems

23 problems



SUR LES

PROBLÈMES FUTURS DES MATHÉMATIQUES,

PAR M. DAVID HILBERT (Göttingen),

TRADUITE PAR M. L. LAUGEL (*).

Qui ne soulèverait volontiers le voile qui nous cache l'avenir afin de jeter un coup d'œil sur les progrès de notre Science et les secrets de son développement ultérieur durant les siècles futurs? Dans ce champ si fécond et si vaste de la Science mathématique, quels seront les buts particuliers que tenteront d'atteindre les guides de la

Hilbert's lecture from the Paris Proceedings

Problem #1 The Continuum Problem

Q: $\boxed{2^{\aleph_0} = \aleph_1}$?

Note: this unique ordinal only exists due to AC.

Cardinal: initial ordinal α s.t. $\forall \beta < \alpha \; \beta \not\sim \alpha$
[$X \sim Y :\Longleftrightarrow$ there is a bij. betw. X & Y]

2^{κ}: unique initial ordinal in bij. with $\mathcal{P}(\kappa)$.

Historical note
Hilbert noted in 1900 that the form. $2^{\aleph_0} = \aleph_1$ implies some fragment of choice.

$$\boxed{2^{\aleph_0} = \aleph_1}$$

[This presupposes that $\wp(\mathbb{N})$ can be wellordered.]

Cantor's theorem:

does not need AC

No surjection from $\mathbb{N}$ onto $\wp(\mathbb{N})$.

Thus: no injection from $\wp(\mathbb{N})$ into $\mathbb{N}$.

With AC, this implies $\aleph_1 \leq 2^{\aleph_0}$.

[An injection from $\aleph_1$ into $\wp(\mathbb{N})$ requires AC.]

Without AC, we get:

Prop. There is a surj. from $\wp(\mathbb{N})$ onto $\aleph_1$.

Proof. By Cantor's zigzag: $\mathbb{N} \sim \mathbb{N} \times \mathbb{N}$.

Thus: $\wp(\mathbb{N}) \sim \wp(\mathbb{N} \times \mathbb{N})$.

I am going to show: $f : \wp(\mathbb{N} \times \mathbb{N}) \longrightarrow \{\alpha \,;\, \omega \leq \alpha < \omega_1\}$ surjection.

Take $R \subseteq \mathbb{N} \times \mathbb{N}$. Consider $(\mathbb{N}, R)$. Define

$$f(R) := \begin{cases} \alpha & \text{if } (\mathbb{N}, R) \cong (\alpha, \in) \\ 0 & \text{o/w.} \end{cases}$$

Claim: f is a surjection. If $\omega \leq \alpha < \omega_1$, there is a bij. $g : \omega \longrightarrow \alpha$. Define

$(n,m) \in R_g \iff g(n) \in g(m)$. $(\alpha, \in) \cong (\mathbb{N}, R_g)$.

q.e.d.

Hilbert #1

$$\underbrace{\boxed{2^{\aleph_0} = \aleph_1}}_{\text{CH}} \; ?$$

Solution to Hilbert #1:

Two parts:

1. CH is consistent — Gödel 1938 (will be proved in this course)
2. $\neg$CH is consistent — Cohen 1962 (will not be proved in this course)

INDEPENDENCE

(the truth value of CH is not determined by ZF).

GRENZEN DES NATURERKENNENS

IGNORABIMUS

Hilbert Radio Address 1930

Wir müssen wissen,
wir werden wissen.

Kongreß deutscher Naturforscher & Ärzte (Königsberg 1930: Hilbert was made Honorary Citizen of Königsberg & gave a national radio address.)

Emil du Bois-Reymond

Born	Emil Heinrich du Bois-Reymond 7 November 1818 Berlin, Kingdom of Prussia
Died	26 December 1896 (aged 78) Berlin, Germany
Nationality	German

Emil: older brother of Paul [mathematician who first defined Cantor's zigzag fn]

Paul David Gustav du Bois-Reymond.

1931: Gödel's Incompleteness Thm shows that there might be an IGNORABIMUS.

CH:

CH is consistent (Gödel 1938)

Main goal of this course (7 or 8 lectures)

INNER MODELS

Thm If $(M, \in) \models ZF$ then there is $\boxed{N \subseteq M}$ s.t. $(N, \in) \models$ ZF + AC + CH.

In fact, this N is going to be the "minimal" inner model.

Kurt Gödel

Gödel around 1926

Born	Kurt Friedrich Gödel April 28, 1906 Brünn, Austria-Hungary
Died	January 14, 1978 (aged 71) Princeton, New Jersey, U.S.

$\neg$CH is consistent (Cohen 1962)

OUTER MODELS
("Forcing")

Thm If $(M, \in) \models ZFC$ which is countable & transitive, then there is $\boxed{N \supseteq M}$ such that

$N \models ZFC + \neg CH$.

This is the topic of a different course.

Paul Cohen

American mathematician

Paul Joseph Cohen was an American mathematician. He is best known for his proofs that the continuum hypothesis and the axiom of choice are independent from Zermelo–Fraenkel set theory, for which he was awarded a Fields Medal. Wikipedia

Born: 2 April 1934, Long Branch, New Jersey, United States

Died: 23 March 2007, Stanford, California, United States

Known for: Cohen forcing; Continuum hypothesis

Fields: Mathematics

§ 2 Substructures

$\mathcal{L}$ first-order language

R relation symbol
f function symbol
c constant symbols

(M, α) $\mathcal{L}$-structure
where $\alpha(R)$ is a relation
$\alpha(f)$ is a function
$\alpha(c) \in M$

$N \subseteq M$ **substructure** if

(a) for every c, $\alpha(c) \in N$

(b) for every f, n-ary, $x_1, \dots, x_n \in N$
$$\alpha(f)(x_1, \dots, x_n) \in N$$

More precisely,

(N, α')

$$\alpha'(c) := \alpha(c)$$
$$\alpha'(f) := \alpha(f) \cap N^n \times N$$
$$\alpha'(R) := \alpha(R) \cap N^n$$

Usually, we write (N, α) for this.

This is literally the notion of substructure that we know from other parts of maths.

Def. Let φ be a formula, (M, α) an α-structure and $N \subseteq M$, we say that

φ is $\begin{cases} \text{upwards absolute} \\ \text{downward absolute} \\ \text{absolute} \end{cases}$ $\begin{array}{|c|} \Rightarrow \\ \Leftarrow \\ \Leftrightarrow \end{array}$ between N & M

n free variable

if $\forall x_1, \dots, x_n \in N$

$(N, \alpha) \models \varphi(x_1, \dots, x_n)$ $\begin{array}{|c|} \Rightarrow \\ \Leftarrow \\ \Leftrightarrow \end{array}$ $(M, \alpha) \models \varphi(x_1, \dots, x_n)$

Def. Atomic formulas $t = t'$ or $R(t_1, \dots, t_n)$

Propositional fmlas: Closure of atomic under $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$

$\exists$-formula if $\exists x \psi(x)$ ψ is prop.

$\forall$-formula $\forall x \psi(x)$ — " —

$\exists\forall$-formula $\exists x \forall y \psi(x,y)$ — " —

$\forall\exists$-formula $\forall x \exists y \psi(x,y)$ — " —

Proposition All atomic & propositional formulas are absolute between N & M if N is a substructure.

Not so for $\exists, \forall$-formulas:

$\mathcal{L}$ language of group theory $+, 0$

$$\exists x \underbrace{\underbrace{(x \neq 0)}_{\text{proposition}}}_{\exists\text{-formula}} \iff : \varphi_{NT}$$

Take $\{0\} \leq \mathbb{Z}$

Have $\{0\} \not\models \varphi_{NT}$ & $\mathbb{Z} \models \varphi_{NT}$.

Similarly: $\forall x (x = 0) \iff : \varphi_T$

$\forall$-formula

$\{0\} \models \varphi_T$ & $\mathbb{Z} \not\models \varphi_T$.

Proposition ① If φ is upwards absolute between N & M, then so is $\exists x \varphi$.

② If φ is downwards absolute betw. N & M, then so is $\forall x \varphi$.

So $\exists$-formulas are upwards & $\forall$-formulas are downw. abs. for substructures.

$\forall\exists$-formulas

$$\forall x \exists y \, (y+y=x) = \varphi_2$$

Everything is divisible by two.

$$\begin{array}{ccccc} \mathbb{Z} & \leq & \mathbb{Q} & \leq & \mathbb{Q}\times\mathbb{Z} \\ \rotatebox{90}{$\models$} & & \rotatebox{90}{$\models$} & & \rotatebox{90}{$\models$} \\ \neg\varphi_2 & & \varphi_2 & & \neg\varphi_2 \end{array}$$

Thus φ_2 is neither upwards nor downwards absolute.

Def. A substructure N is called an elementary substructure if all formulas are absolute.

$$N \preccurlyeq M$$

Reminder The isomorphism lemma states: if $f : (N,\alpha') \cong (M,\alpha)$ is an isomorphism, then $\{\varphi ; \varphi \text{ is sentence } \& \ N \models \varphi\} = \{\varphi ; \varphi \text{ is sentence } \& \ M \models \varphi\}$

$\mathrm{Th}(N) =$ ... $=: \mathrm{Th}(M)$

We say that N & M are elementarily equivalent.

In symbols: $N \equiv M$.

The proof is a proof by induction & proves

$$(N,\alpha')\models\varphi(x_1,\dots,x_n)\iff(M,\alpha)\models\varphi(f(x_1),\dots,f(x_n))$$

This property is called: f is an elementary embedding

Thus: $N\preccurlyeq M\iff \mathrm{id}:N\longrightarrow M$ is an elementary embedding.

§3 The language of set theory

$$\mathcal{L}_\in = \{\in\}$$

So, no constant or function symbols. That means: no nontrivial atomic formulas and thus almost nothing is absolute.

$$x=0 \Leftrightarrow x=\emptyset \;:\Longleftrightarrow\; \forall z\,(z\notin x) \Longleftrightarrow \Phi_0(x)$$

$\forall$-formula

So, not necessarily absolute.

$$x=1\Longleftrightarrow x=\{\emptyset\} \;:\Longleftrightarrow\; \forall z\,(z\in x\Longleftrightarrow\Phi_0(z))$$

$$\Longleftrightarrow \Phi_1(x)$$

<u>Example</u> Let (M, E) be any $\mathcal{L}_\in$-structure

with $(M, E) \models \Phi_0(m_0)$

$(M, E) \models \Phi_1(m_1)$

Form $M' := M \setminus \{m_0\}$

$$(M', E) \models \Phi_0(m_1)$$

$$\forall z\,(z \notin m_1)$$

$$(M', E) \models \neg\Phi_1(m_1)$$

Thus: Very basic properties of set theory are not preserved by $\mathcal{L}_\in$-sub-structures.

<u>Message</u> Don't expect a formula to mean what you think it means.

Example $(M, \in) \models ZFC$

This example illustrates how a notion like "power set" is not fixed by the axioms but will be re-interpreted in each model.

$\mathbb{N} \subseteq M$

$\{0, 1, 2, 3, \ldots\}$

$\emptyset \quad \{\emptyset\} \quad \{\emptyset, \{\emptyset\}\} \quad \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

Consider $(\mathbb{N}, \in)$.

Which axioms of ZF does $(\mathbb{N}, \in)$ satisfy?

for instance $\exists x\, \Phi_0(x)$

$$\forall x \forall y \exists u \forall z (z \in u \Longleftrightarrow z \in x \vee z \in y)$$

What about power set?

$$\forall x \exists p \forall z (z \in p \Longleftrightarrow \forall w (w \in z \Rightarrow w \in x))$$

If $k, n \in \mathbb{N}$, when is $k \subseteq n$?

iff $k \leq n$.

So the "$(\mathbb{N})$-power set" of n is

$\{k ; k \leq n\} = n+1$.

So $(\mathbb{N}, \in) \models$ Power set

and powersets do not always have card. 2^k.

Def. If (M, E) is an $\mathcal{L}_\in$-structure, we say that $N \subseteq M$ is a transitive substructure if for all $x \in N$ & $y \in M$, if yEx, then $y \in N$.

[That's just a model-theoretic reformulation of the standard notion of "transitive set".].

If $N \subseteq M$ is transitive, then Φ_0 is absolute between N & M.

Proof. Since Φ_0 was $\forall$-formula, it is downwards absolute. Thus only need to show upwards.

Suppose not: $(N, E) \models \Phi_0(x)$ & $\underbrace{(M, E) \models \neg\Phi_0(x)}$

$\Downarrow$

there is $y \in M$ s.t. yEx

$y \in N$

$\Longrightarrow (N, E) \models \neg\Phi_0(x)$.

by transitivity

Contradiction! q.e.d.