XI ELEVENTH LECTURE

Computability, Decidability, Incompleteness

27 June 2023

Last time (Lecture X)

The **Entscheidungsproblem** is undecidable (= not computable).

Proved this by $\mathbb{K} \leq_m \{\varphi ; \vdash \varphi\}$

Clarification

Lecture X Page 9

$$\exists n\ \mathrm{NAT}(n) \wedge \mathrm{CSTATE}(\mathrm{COMP}(M, x, n)) = \mathrm{HALT}$$

$$\overset{!!}{\psi(M, x)}$$

ψ describes "M halts at input x".

Remember we had $\psi_w(x)$ describing "x is the seq. $\vec{w}$ where $\mathrm{code}(\vec{w}) = w$".

Now specialise to $\psi^{*}_{w}(x)$ describing "x is the seq. of length 1 with unique element w".

Let Φ be the conjunction of all of our axioms 1. to 27.

$$\boxed{\sigma_w := \forall M \forall x\ \Phi \longrightarrow [\varphi_w(M) \wedge \psi^{*}_{w}(x) \wedge \psi(M, x)]}$$

Reminder Encoding of formulas as finite sequences:

In order for

$$\{\varphi;\ \vdash\varphi\} \subseteq \mathbb{W},$$

we need to make sure that all symbols in φ occur as symbols in Σ.

So if the logical and non-logical symbols of $\mathcal{L}$ are all in Σ, formulas are words in $\mathbb{W}$. [Not just represented by.]

Problem Logical languages have infinitely many variables and no bound whatsoever on the number of nonlogical symbols.

Solution 1 If we show that the ~~Entscheidungs~~- problem is unsolvable for any finite restriction of $\mathcal{L}$, then any bigger problem cannot be decidable either.
[This would only work if the proof works with some finite fragment.]

Note: Our proof will work with finitely many non-logical symbols, but might use unboundedly many variables.

Solution 2 [Remember the similar problem when we encoded machines as words and we needed infinitely many symbols for states.]

Use 0, 1 and encode variable in binary, i.e.

V0 $\longrightarrow$ stands for v_0

V1 $\longrightarrow$ v_1

V10 $\longrightarrow$ v_2

V11 $\longrightarrow$ v_3

etc.

Summary We shall make Σ big enough to include 0, 1, all logical constants $(\exists, \forall, \wedge, \vee, \neg, (, ,,))$, and a finite number of non-logical symbols to be determined in the proof.

Lecture IX, pages 8-9

All of the relevant objects: sequences, sequences of sequences, functions etc live in a small set theoretic universe: HF, the set of hereditarily finite sets:

$$V_0 := \emptyset$$

$$V_{i+1} := \mathcal{P}(V_i)$$

$$HF := \bigcup_{i\in\mathbb{N}} V_i$$

Once we have identified the elements of Σ with fixed elements of HF, all terminology used in computability

word
configuration
instruction
state
⋮

have concrete meanings in HF.
In particular, statements such as

$$f_{v_0,1}(v)\downarrow$$

have a concrete meaning in HF via the descriptions given in section D of the course.

REMINDER

FST — Finite Set Theory — Pairing, Union, Powerset, Separation, Foundation

Zermelo Z — FST + Infinity

Zermelo-Fraenkel ZF — Z + Replacement

① $HF \models FST + \neg\,\text{Infinity} + \text{Replacement}$

② $\text{Ord} \cap HF = \mathbb{N}$

PART I : INCOMPLETENESS

We start with an extensive review of what we know about logic.

Important remark:

The term "completeness" in the Completeness Thm & the Incompleteness Thm mean different things.

In Completeness Thm **completeness** is a property of $\vdash$ (the relation "is provable")

We say that a provability predicate $\vdash$ is complete if

$$\{\varphi ; T \vdash \varphi\} = \{\varphi ; T \models \varphi\}$$

for any T.

The Completeness Thm for first-order logic states that the standard provability predicate $\vdash$ has this property.

Because for the standard $\vdash$, proofs are finite objects, this means $T \vdash \varphi \Longleftrightarrow \exists$ finite $T_0 \subseteq T$ $T_0 \vdash \varphi$.

COROLLARY

Compactness Thm

$T \models \varphi \Longleftrightarrow \exists$ finite $T_0 \subseteq T$ $T_0 \models \varphi$.

equivalently

T has a model $\Longleftrightarrow$ every finite $T_0 \subseteq T$ has a model

Consequences of Compactness

Infinite structures cannot be classified up to isomorphism by first-order logic.

i.e., if M is infinite, then there is no set of sentences S s.t. ⊛

$\forall N \quad N \cong M \iff N \models S$.

Why does this follow from compactness?

UPWARDS LÖWENHEIM-SKOLEM

For every $M \models S$ where M is infinite and every set X, there is $M^* \models S$ s.t. X injects into M^*.

This implies ⊛ : if $|X| > |M|$, then $|M| < |X| \leq |M^*|$, so $M \not\cong M^*$.

[Proof sketch of Upw. L-S:
Add constant symbol c_x for every $x \in X$ and consider $S^* := S \cup \{c_x \neq c_{x'} ; x \neq x'\}$.
If $S_0 \subseteq S^*$ is finite, then it contains finitely many of the new constant symbols, so in the infinite structure M, we find enough diff. objects to make all statements $c_x \neq c_{x'}$ in S_0 true.
Thus this expansion of M is a model of S_0.
By compactness S^* has a model.]

Def. If M is a structure, we write

$$\mathrm{Th}(M) := \{\varphi ;\ M \models \varphi\}$$

the ~~sentences~~ sentences true in M

and say $M \equiv N$

M and N are elementarily equivalent

$\iff \mathrm{Th}(M) = \mathrm{Th}(N)$.

In the above argument, it is possible that while $M^* \not\cong M$, we can still have

$M^* \equiv M$.

[In our Upw. L-S proof, let $S := \mathrm{Th}(M)$. Then $M^* \models S^* \supseteq \mathrm{Th}(M)$, so $M \equiv M^*$.]

Question Is it possible for some M s.t. there is S characterising M up to elementary equivalence, i.e.,

TRIVIALLY YES (next page)

$$\forall N \quad N \models S \overset{*}{\iff} N \models \mathrm{Th}(M) \iff M \equiv N.$$

*: This is due to a special property of $\mathrm{Th}(M)$.

Definition A set S of formulas is called (negation-) complete if for all φ, either $S \vdash \varphi$ or $S \vdash \neg\varphi$.

NEGATIONS-TREUE (EFT)

Thus if $M \models S$.

① S is negation-complete $\iff S \vdash Th(M)$

[S & $Th(M)$ are logically equivalent]

② S is negation-complete $\iff$ all models of S are elementarily eq.

③ S is negation-complete

$$\iff \underbrace{\{\varphi ; S \vdash \varphi\}}_{\text{PROVABILITY}} = \{\varphi ; \underbrace{M \models \varphi}_{\text{TRUTH}}\}$$

Provability & truth coincide.

→ Therefore, the answer to our question from page 6 is

YES

$Th(M)$ characterises M up to elementary equivalence.

IMPROVED QUESTION:

Is there a decidable S s.t. S characterises M up to elementary equivalence.

By our observations on page 7, this is equivalent to:

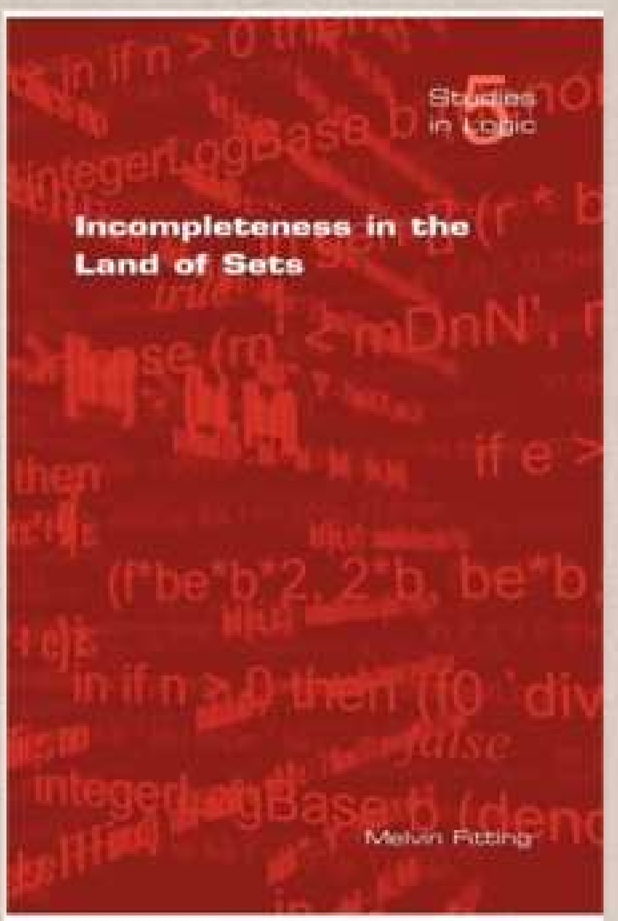
Is there is decidable negation-complete S s.t. $M \models S$?

Gödel's Incompleteness Thm gives a negative answer for a concrete model M.

Traditionally $M = \mathbb{N}$.
often formulated as "PA [the standard axioms for $\mathbb{N}$] is not negation-complete".

Our formulation is called ESSENTIAL INCOMPLETENESS: not only some axiom system is not negation-complete, but there is none.

Instead of doing this in $\mathbb{N}$ with theory PA (or $Th(\mathbb{N})$), we move to HF following the lead of



MELVIN FITTING

We saw $HF \models \boxed{FST + \neg \text{Infinite}} =: HFST$

By what we just discussed, HFST cannot characterise HF up to isomorphism.

[Using upw. L-S: HF is countable, take any uncountable X, say, $\mathbb{R} =: X$ and get a model of HFST of size $\geq |\mathbb{R}|$; clearly not isomorphic to HF.]

<u>Aside</u>. In this particular case, we can get even a countable non-isomorphic model:

Let c be an additional constant symbol and

$\varphi_n :=$ c is an ordinal and there are at least n ordinals smaller than c.

Consider $S^* := HFST \cup \{\varphi_n ; n \in \mathbb{N}\}$

If $S_0 \subseteq S^*$ finite, it only contains finitely many of the φ_n. Let N be big enough that $n < N$ for all n s.t. φ_n occurs in S_0. Then interpreting c by N in HF means

$$(HF, N) \models S_0 .$$

Since S_0 was arbitrary, by compactness, there is a model $M^* \models S^*$.

But then c^{M^*} is an ordinal that has ∞ many ordinals below; such a thing doesn't exist in HF, so $HF \not\cong M^*$.

But the term model (as in the proof of completeness) is countable.

Gödel's Incompleteness Theorem
in the land of sets

There is no decidable negation-complete S s.t. $HF \models S$.

By the remarks on page 7, we only need to show that

$$P := \{\varphi ; S \vdash \varphi\} \neq \{\varphi ; HF \models \varphi\} =: T$$

PROVABILITY TRUTH

This assumption *must* be necessary. (*)

We'll do this in two steps:

(1) If S is decidable, then $\{\varphi ; S \vdash \varphi\}$ is Σ_1.

(2) $\{\varphi ; HF \models \varphi\}$ is Π_1-hard.

Note that since $\Sigma_1 \neq \Delta_1$ [by Turing's Halting Theorem], we find some $A \in \Pi_1 \setminus \Sigma_1$. If $P = T$, then by (2) P is Π_1-hard, so $A \leq_m P$. But by (1) P is Σ_1, so A is Σ_1.

Contradiction!

(*) Clearly if $S := T$. Then S is negation-complete and thus $P = T = S$. But we will not be able to prove that P is Σ_1.

Step 1 Prove that if S is decidable, then $P := \{\varphi ; S \vdash \varphi\}$ is Σ_1.

REMINDER How did we define the standard proof predicate $\vdash$?

In MLML, we followed EFT (Ebbinghaus, Flum, & Thomas) and used Gentzen's sequent calculus.

Here, we shall assume that we have a natural deduction system:

SLOGAN A proof is a sequence of formulas where each formula is either an axiom or follows from previous ones by a rule.

A set $R \subseteq \mathrm{Fml}^{n+1}$ is called an **n-ary rule** and we interpret

$(\varphi_0, \ldots, \varphi_n) \in R$ as

if $\varphi_0, \ldots, \varphi_{n-1}$ have been derived, then φ_n can be derived.

Since $\mathrm{Fml} \subseteq \mathbb{W}$, rules are sets of sequences of words, which we know how to encode as words.

A rule R is **computable** if it is computable as a set of words.

A **calculus** is a finite set $\mathcal{R}$ of rules and it's called **computable** if all it's rules are computable.

OBSERVE (by inspection of the definitions)
The standard proof predicate $\vdash$ has a computable calculus.

Goal for next time:

If S is computable and $\mathfrak{R}$ is computable, then

$$\{\varphi\, ;\, S \vdash_{\mathfrak{R}} \varphi\}$$

is Σ_1.

If we've proved that, Step 1 is complete.