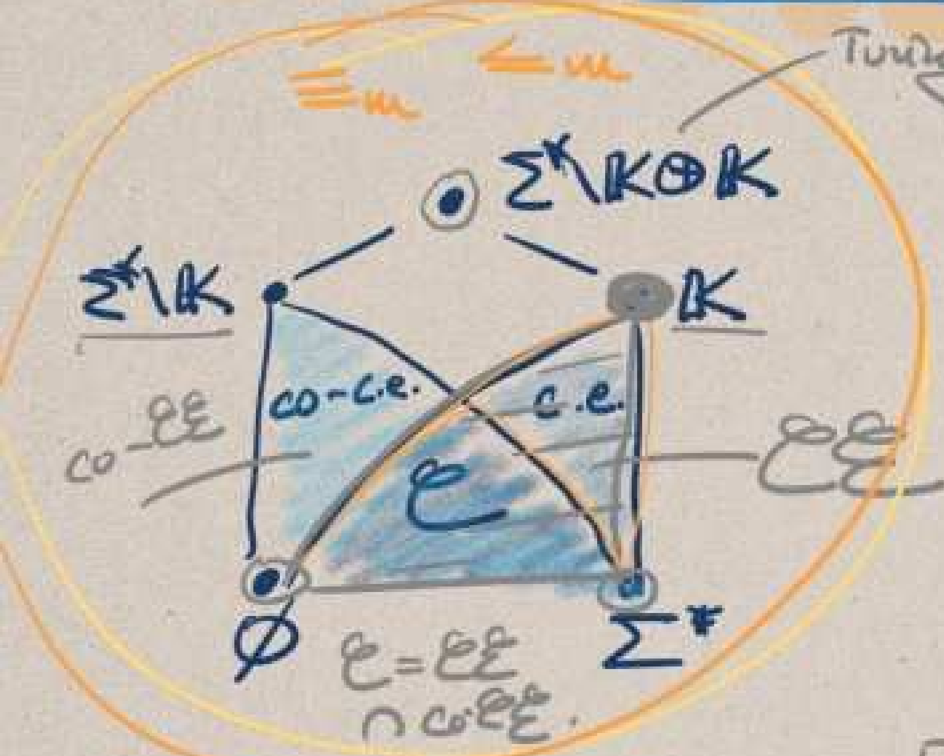
RECURSION THEORY

VIII

5 January 2022



A is an INDEX SET if there is a $\underline{Z \subseteq \mathcal{EE}}$ s.t.

$$A = \{ P ; W_P \in Z \}$$

$$\boxed{W_P = \operatorname{dom}(f_P)}$$

[A is trivial if $Z = \emptyset$ or $Z = \mathcal{EE}$]

<u>Rice's Theorem</u> [INDEX SET THEOREM]

if A is a non-trivial index set then A is not computable.

<u>More precisely</u>:

if $\emptyset \in Z$, then $\underline{\Sigma^*\setminus\mathbb{K}} \leq_m A$.

if $\emptyset \notin Z$, then $\mathbb{K} \leq_m A$

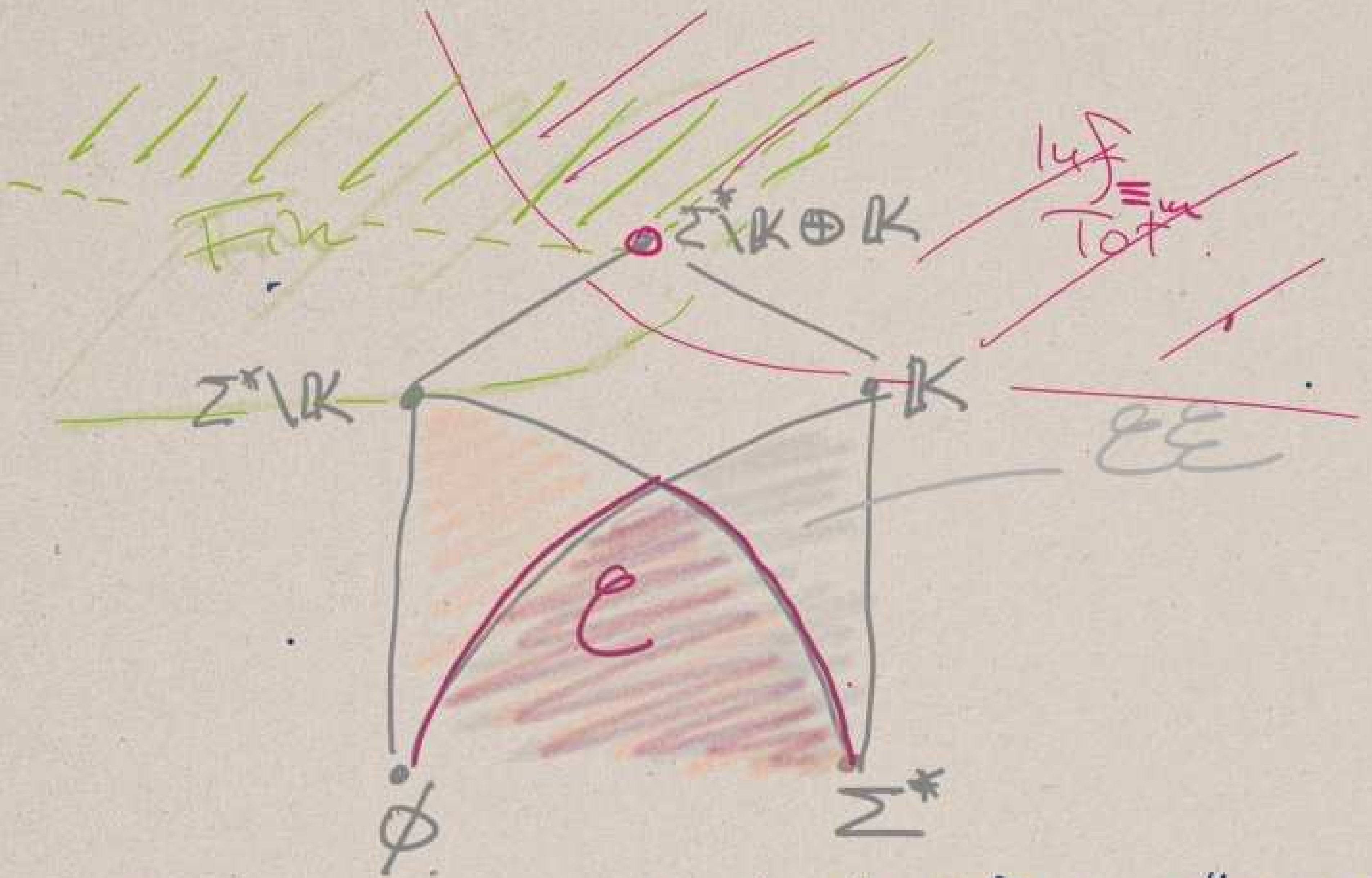
<u>Examples</u> of <u>nontrivial</u> <u>index sets</u>

$\mathbb{K} \equiv_m \underline{\mathbb{K}_1} =:$ Nonemp $= \{ P ; W_P \neq \emptyset \}$

Tot $= \{ P ; W_P = \Sigma^* \}$

Fin $= \{ P ; W_P \text{ finite} \}$

Inf $= \{ P ; W_P \text{ infinite} \}$



Today's lecture is about identifying the position of Fin, Inf, Tot in this picture.

<u>Corollary</u> (to the proof of Rice's Theorem).

(1) $\Sigma^* \setminus \mathbb{K} \leq_m \mathrm{Fin}$

(2) $\mathbb{K} \leq_m \mathrm{Inf}$

(3) $\mathbb{K} \leq_m \mathrm{Tot}$

Pf. Follows directly from the fact that $\emptyset$ is finite, not infinite and not $= \Sigma^*$. q.e.d.

Theorem $\mathrm{Inf} \equiv_m \mathrm{Tot}$.

Proof. We split the proof in two parts:

I. $\mathrm{Tot} \le_m \mathrm{Inf}$.

II. $\mathrm{Inf} \le_m \mathrm{Tot}$.

Part I. $\mathrm{Tot} \le_m \mathrm{Inf}$.

$$g : \Sigma^* \times \Sigma^* \longrightarrow \Sigma^*$$

Reminder On Σ^*, we had a canonical ordering of order type $\mathbb{N}$: start with the empty word, then words of length one, then words of length two, lexicographically, and so on.

[Since Σ is finite, this ordering has order type $\omega = \mathbb{N}$.]

We are writing $u < v$ for words $u, v \in \Sigma^*$ for this ordering.

$$g(w,v) := \begin{cases} f_w(v) & \text{if } f_w(u)\downarrow \text{ for all } u \le v \\ \uparrow & \text{o/w} \end{cases}$$

$$g(w,v) := \begin{cases} f_w(v) & \text{if } f_w(u)\downarrow \text{ for all } u \le v \\ \uparrow & \text{o/w} \end{cases}$$

Case 1

Case 2

g is a computable function

$$w * v \longmapsto g(w,v),$$

therefore by the s-m-n theorem there is a total computable $h : \Sigma^* \longrightarrow \Sigma^*$ s.t.

$$f_{h(w)}(v) = g(w,v)$$

<u>Claim</u> h is a many-one reduction from Tot to Inf.

1. Let $w \in$ Tot. So f_w is a total function. So in the above case distinction we're always in Case 1.
   So $g(w,v)\downarrow = f_w(v)$.
   Thus $f_{h(w)}$ is total.
   $h(w) \in \text{Tot} \subseteq \text{Inf}$.

Have proved $w \in \text{Tot} \Rightarrow h(w) \in \text{Inf}$.

NOTE THAT WE PROVED SOMETHING STRONGER, WE'LL REMARK ON THIS LATER.

2. Show that

$$w \notin Tot \implies h(w) \notin Inf.$$

If $w \notin Tot$. So there is some $u \in \Sigma^*$ s.t. $f_w(u)\uparrow$.

$$g(w,v) = \begin{cases} f_w(v) & \text{if } f_w(u)\downarrow \ \forall u \leq v \\ \uparrow & \text{o/w} \end{cases}$$

Therefore, if $v \geq u$, we are not in Case 1 of the case destinction, but in Case 2. Therefore $f_{h(w)}$ is only defined for words $v < u$. Thus $W_{h(w)}$ is finite and therefore $h(w) \in Fin$

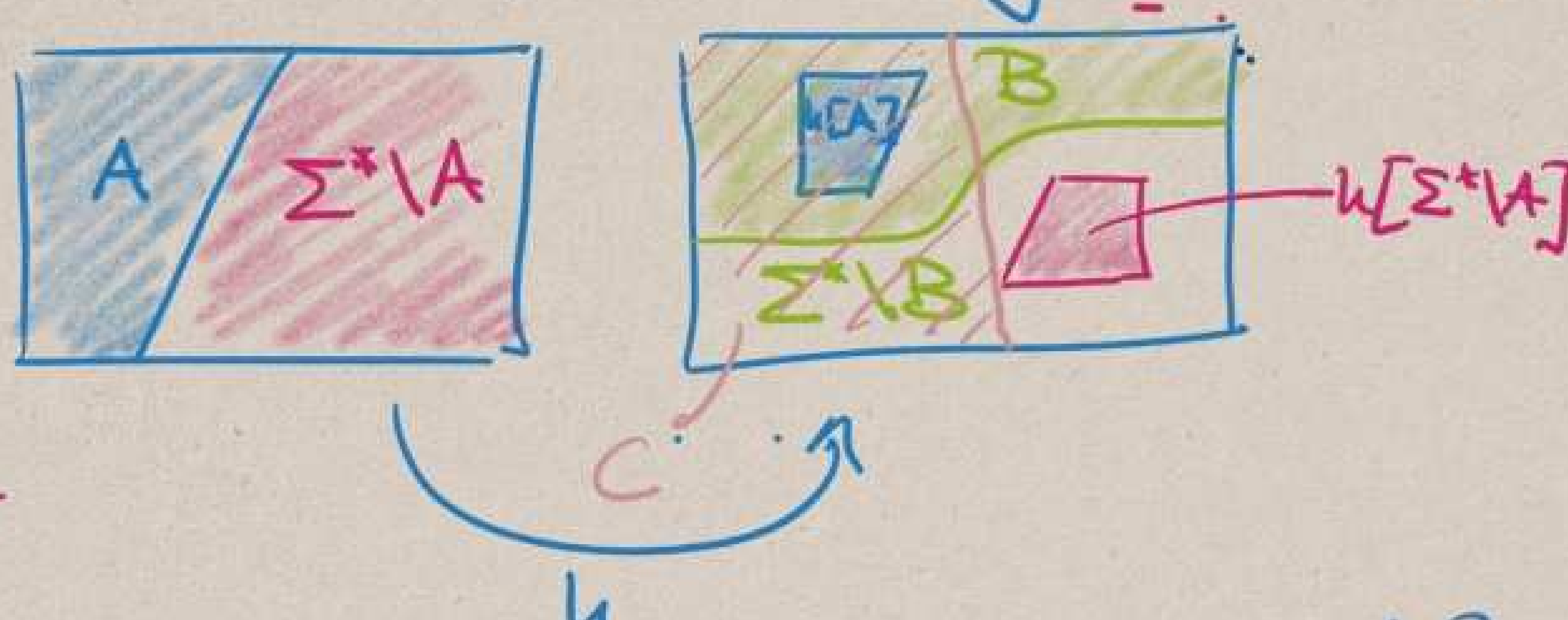
$$\implies h(w) \notin Inf.$$

Together, 1. & 2. show that h is a reduction from Tot to Inf.

This proves Part I of our theorem.

REMARK.

If Σ^* is represented by a rectangle



with $h[A] := \{h(w) ; w \in A\}$

$h[\Sigma^* \setminus A] := \{h(w) ; w \notin A\}$

Function h is a reduction from A to B if the picture looks as above:

- $h[A]$ and $h[\Sigma^* \setminus A]$ are disjoint
- B separates these two sets.

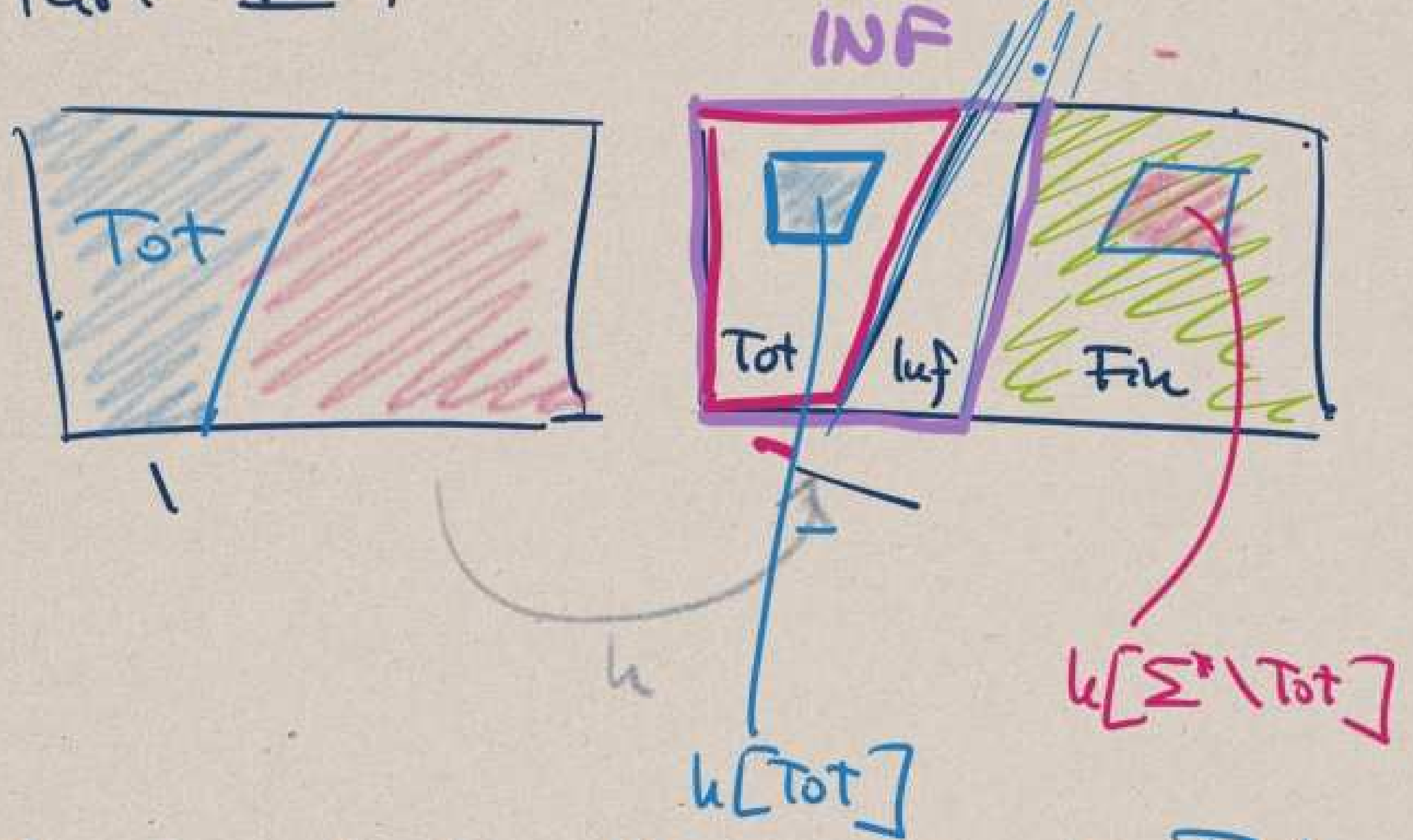
But B is certainly not unique with this property (unless h was a surjection)

Every C s.t. $h[A] \subseteq C$ and

$h[\Sigma^* \setminus A] \cap C = \emptyset$

has the property that h is a red. from $A \leq_m C$.

Apply this remark to the proof of Part I:



So the dividing lines given by Tot and Inf both yield the reduction property for h.

$$\mathrm{Tot} \leq_m \mathrm{Tot}$$ this one is not interesting

$$\mathrm{Tot} \leq_m \mathrm{Inf}$$

Moreover if X is s.t. $\mathrm{Tot} \subseteq X \subseteq \mathrm{Inf}$, this proof shows that $\mathrm{Tot} \leq_m X$.

Back to our proof of the theorem:

Part II. $\mathrm{Inf} \leq_m \mathrm{Tot}$.

$$g(w,v) := \begin{cases} f_w(u) & \text{where } u \text{ is the smallest } u \geq v \text{ s.t. } f_w(u)\downarrow, \text{ if it exists} \\ \uparrow & \text{o/w.} \end{cases}$$

We use our technique of diagonalisation or zig-zag to show that g is computable:
List all words $\geq v$ in order type ω:

$$v = w_0, w_1, w_2, w_3, \ldots$$

s.t. $\{w_i \,;\, i \in \mathbb{N}\} = \{u \in \Sigma^* \,;\, u \geq v\}$

At step k, consider $k = \langle i, j \rangle$ and run the computation with input w_i for j steps. If it halts, halt, otherwise continue with $k+1$.

So g is computable: $w * v \longmapsto g(w,v)$

So by the s-m-n theorem, find total computable h s.t. $f_{h(w)}(v) = g(w,v)$.

<u>Claim</u> h reduces Inf to Tot.

$$f_{h(w)}(v) = \begin{cases} f_w(u) & u \text{ is the least } u \geq v \text{ s.t. } f_w(u)\downarrow \\ \uparrow & \text{o/w} \end{cases}$$

1. Case 1.
   $w \in \text{Inf}$.
   $\Longrightarrow f_w$ halts for infinitely many inputs; i.e., for each fixed v, it'll halt for some $u \geq v$.
   So $f_{h(w)}$ will always halt, so $f_{h(w)}$ is total. And thus $h(w) \in \text{Tot}$.
2. Case 2. $w \notin \text{Inf}$.
   $\Longrightarrow f_w$ only halts for finitely many inputs. Take v s.t. f_w does not halt for any $u \geq v$. By definition, $f_{h(w)}(v) \uparrow$. So $f_{h(w)}$ is not total, so $h(w) \notin \text{Tot}$.

This finishes Part II and proves the thm. q.e.d.

<u>Remark</u> An analysis of the proof of Part II shows that

$$h[\Sigma^* \setminus \mathrm{Inf}] \subseteq \mathrm{Fin}.$$

So if $\mathrm{Fin} \subseteq X \subseteq \Sigma^* \setminus \mathrm{Tot}$, then

$$\mathrm{Inf} \leq_m X.$$

<u>Theorem</u> $\mathbb{K} \leq_m \mathrm{Fin}$.

<u>Corollary</u> This implies that

$$\mathbb{K} \oplus \Sigma^* \setminus \mathbb{K} \leq_m \mathrm{Fin}, \mathrm{Inf}, \mathrm{Tot}.$$

<u>Proof.</u> We get $\mathbb{K}, \Sigma^* \setminus \mathbb{K} \leq_m \mathrm{Fin}$. Since the Turing join is the least upper bound, we get that $\mathbb{K} \oplus \Sigma^* \setminus \mathbb{K} \leq_m \mathrm{Fin}$.

We observed that

$$\Sigma^* \setminus (\mathbb{K} \oplus \Sigma^* \setminus \mathbb{K}) \equiv_m \mathbb{K} \oplus \Sigma^* \setminus \mathbb{K}$$

and therefore

$$\mathrm{Inf} = \Sigma^* \setminus \mathrm{Fin} \geq_m \Sigma^* \setminus (\mathbb{K} \oplus \Sigma^* \setminus \mathbb{K})$$

$$\equiv_m \mathbb{K} \oplus \Sigma^* \setminus \mathbb{K}.$$

By the last thm, $\mathrm{Inf} \equiv_m \mathrm{Tot}$, so $\mathbb{K} \oplus \Sigma^* \setminus \mathbb{K} \leq_m \mathrm{Tot}$.

q.e.d.

Proof of Thm $\mathbb{K} \leq_m \mathrm{Fin}$.

$$g(w,v) := \begin{cases} \varepsilon & \text{if } f_w(w) \text{ has not yet halted after } |v| \text{ many steps} \\ \uparrow & \text{o/w} \end{cases}$$

g, i.e., the function

$$w * v \longmapsto g(w,v)$$

is a computable function. Therefore, by the s-m-n Theorem, there is a total computable function h s.t.

$$f_{h(w)}(v) = g(w,v).$$

Claim h reduces $\mathbb{K}$ to Fin.

1. $w \in \mathbb{K}$. This means that $f_w(w)\downarrow$, so there some $n \in \mathbb{N}$ s.t. this computation has halted after n steps.

So for any v s.t. $|v| \geq n$, $g(w,v)\uparrow$.

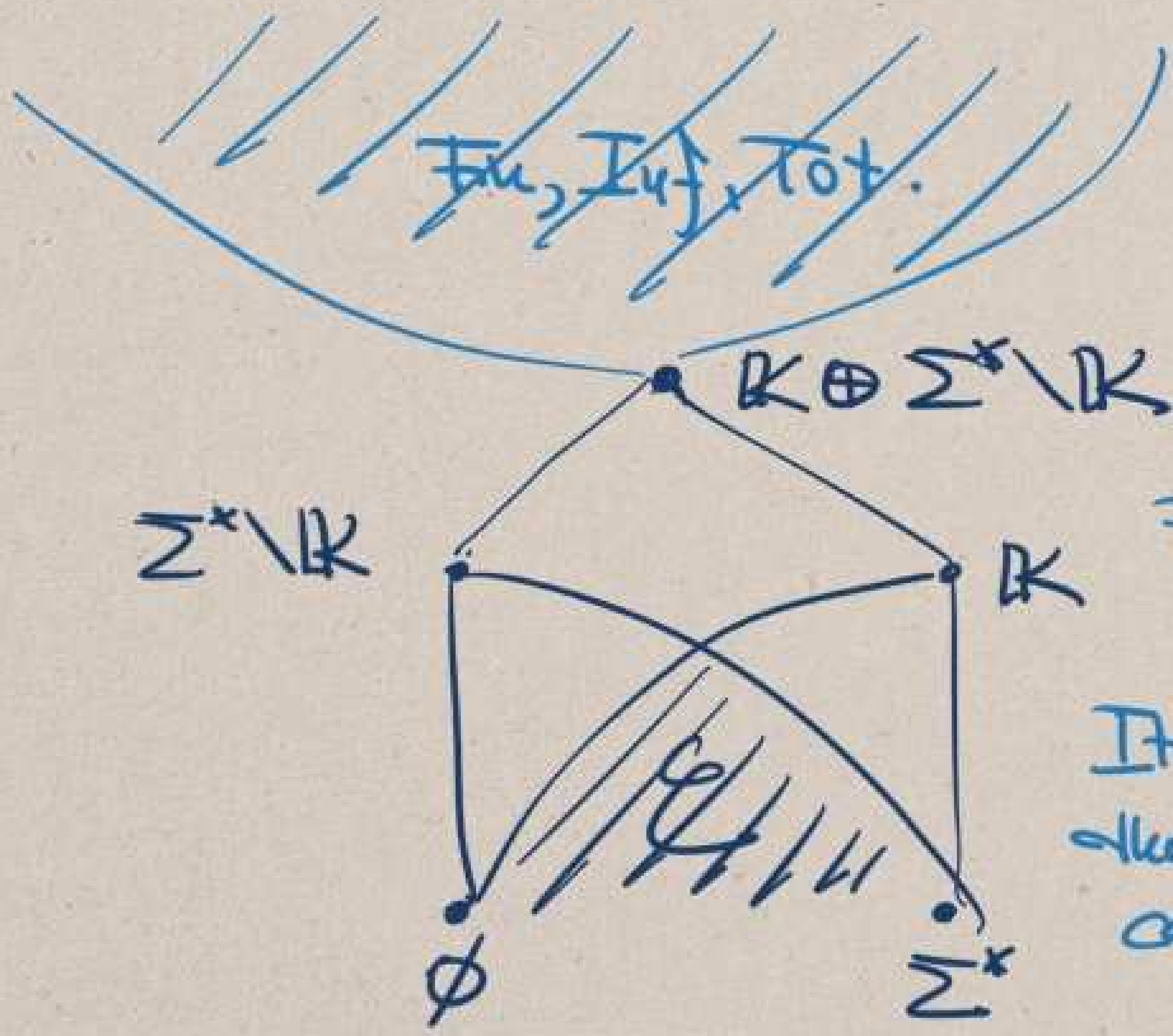
Therefore $f_{h(w)}$ is only defined for finitely many words: $h(w) \in \mathrm{Fin}$.

2. $w \notin \mathbb{K}$.
So $f_w(w)\uparrow$.
Therefore, for all v $f_{h(w)}(v) = \varepsilon$.

$$f_{h(w)}(v) = \begin{cases} \varepsilon & \text{if } f_w(w) \text{ has not yet halted after } |v| \text{ steps} \\ \uparrow & \text{o/w} \end{cases}$$

So $f_{h(w)}$ is total, and therefore
$h(w) \in \mathrm{Tot}$.
$\Longrightarrow h(w) \notin \mathrm{Fin}$.

q.e.d.



So far, it would still be possible that
$\mathrm{Fin} \equiv_m \mathrm{Inf} \equiv_m \mathrm{Tot} \equiv_m \mathbb{K} \oplus \Sigma^* \setminus \mathbb{K}$.

It turns out that this is not the case.

One of the next goals is to show that

$$\left.\begin{array}{l} \mathrm{Inf} \not\leq_m \mathrm{Fin} \\ \mathrm{Fin} \not\leq_m \mathrm{Inf} \end{array}\right] (*).$$

and therefore $\mathrm{Inf}, \mathrm{Fin} \not\leq_m \mathbb{K} \oplus \Sigma^* \setminus \mathbb{K}$.

[Remark: The "therefore" follows from (*) since $\mathbb{K} \oplus \Sigma^* \setminus \mathbb{K}$ is selfdual, and anything $\equiv_m$ to something selfdual must be selfdual.
But $\mathrm{Inf} = \Sigma^* \setminus \mathrm{Fin}$.
So (*) says that Inf is not selfdual.]

A first step in this direction.

Definition A set $A \subseteq \Sigma^*$ is called

- Π_2 if there is a computable set R s.t. $w \in A \longleftrightarrow \forall v \exists u\ (w * v) * u \in R$
- Σ_2 if there is a computable set R $w \in A \longleftrightarrow \exists v \forall u\ (w * v) * u \in R$.

Observation A is Π_2 iff $\Sigma^* \setminus A$ is Σ_2.

$$w \in A \iff \forall v \, \exists u \; (w * v) * u \in R$$
$$w \notin A \iff \neg \forall v \, \exists u \; (w * v) * u \in R$$
$$\iff \exists v \, \forall u \; \neg (w * v) * u \in R$$
$$\iff \exists v \, \forall u \; (w * v) * u \notin R$$

So for the computable set
$$\overline{R} := \{ w \in \Sigma^* ; \; w \notin R \},$$
we have
$$w \notin A \iff \underbrace{\exists v \, \forall u \; (w * v) * u \in \overline{R}}_{\Sigma_2}$$

Reminder A set P is called $\overset{\Sigma_2}{\Pi_2}$-hard if for every $\overset{\Sigma_2}{\Pi_2}$ set A, we have
$$A \leq_m P.$$
A set is called Π_2-complete if it is Π_2-hard and Π_2 itself.

<u>Observation</u> Inf, Tot are Π_2

Fin is Σ_2.

[Let's see this for Tot: _____________ :

$w \in Tot \iff \forall v \, \exists u \; \underbrace{\boxed{f_w(v) \text{ halts after } |u| \text{ steps}}}$

computable.]

<u>Lemma</u> If $A \leq_m B$ and B is Π_2, then so is A.

<u>Pf.</u> Suppose B is Π_2

$w \in B \iff \forall v \, \exists u \, (w * v) * u \in R$

and h reduces A to B.

$w \in A \iff h(w) \in B$

$\iff \forall v \, \exists u \, \underbrace{(h(w) * v) * u) \in R}$

$R' := \{ (h(w) * v) * u ; (w * v) * u) \in R \}$
is computable.

q.e.d.

Thm Tot is Π_2-complete

Pf. Let P be an arbitrary Π_2 set, i.e.,

$$w \in P \iff \boxed{\forall v \exists u \ (w*v)*u \in R.}$$

Define

$$g(w,v) := \begin{cases} \varepsilon & \text{search for } u \text{ s.t. } (w*v)*u \in R \\ \uparrow & \text{o/w} \end{cases}$$

This computable by zig-zag method.

Therefore by s-m-n, we have total computable h s.t.

$$f_{h(w)}(v) = g(w,v).$$

If $w \in P \iff \forall v \exists u \ (w*v)*u \in R$

$\implies f_{h(w)}$ is the constant fn ε

$\implies h(w) \in \mathrm{Tot}$.

If $w \notin P \implies$ there is v s.t. for no u, $(w*v)*u \in R$, thus $f_{h(w)}(v)\uparrow$, so $h(w) \notin \mathrm{Tot}$.

Thus $P \leq_m \mathrm{Tot}$.

So Tot is Π_2-hard.

Since Tot is Π_2, it is Π_2-complete.

q.e.d.

We shall see later (Post's Theorem) that Π_2-complete sets cannot be Σ_2. This implies

$$\mathrm{Inf} \not\leq_m \mathrm{Fin}.$$