# Recursion Theory VI

30 November 2021

## DIAGONALISATION / ZIG-ZAG

**Theorem 1** The intersection of finitely many c.e. sets is c.e.

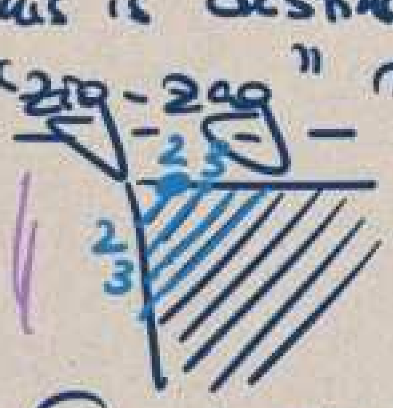
**Theorem 2** The set
$$\{P \,;\, P \text{ is a program s.t. } \operatorname{dom}(f_P) \neq \emptyset\} \text{ is c.e.}$$

"Diagonalisation" often refers to the proof that takes a list and produces a new object that disagrees on the diagonal:

- Cantor's uncountability of $\mathbb{R}$

- Halting problem

This is distinct from the "zig-zag" method:




- Cantor's countability of $\mathbb{Q}$

**NEXT LECTURES**

VII : Tuesday 21 Dec 11–13

VIII : Wednesday 5 Jan 16–18 (AUSWEICHTERMIN 14–16)

IX : Thursday 6 Jan 16–18

X : Wednesday 12 Jan 16–18 (AUSWEICHTERMIN 14–16)

XI : Thursday 13 Jan 16–18

**Theorem 3** Fix $k \in \mathbb{N}$. Then the set

$$A_k := \{ P \,;\, P \text{ is a program s.t. } |\mathrm{dom}(f_P)| \geq k \} \quad \text{is c.e.}$$

**Proof.** We set a flag $:= 0$

storage $:= \emptyset$.

We run an algorithm on input $P$ in (potentially) infinitely many steps.

**Step $n$.** $n = \langle i, j \rangle$

Run $f_P(w_i)$ for $j$ steps.

If it hasn't halted, go to step $n+1$.

If it has halted, check whether $w_i$ is in the storage.

If yes, go to step $n+1$.

If no, then flag $:=$ flag $+ 1$

storage $:=$ storage $\cup \{w_i\}$.

Check whether flag $\geq k$.

If yes, halt and output 1.

O/w go to step $n+1$.

This describes $\varphi_{A_k}$. Therefore $A_k$ is c.e. q.e.d

Remember that we still need to argue that c.e. set $A$ are "computably enumerable".

This means: "enumerated by computable function".

i.e., there is a function $f$ computable s.t. $A$ is listed by $f$.

[Looks like $A = \mathrm{ran}(f)$, but that's already known.]

We mean instead: $\mathrm{ran}(f)$ for a total computable function!

Theorem 4 A set $A \neq \emptyset$ is c.e. iff $A$ is the range of a total computable function.

[Remark. $\emptyset$ cannot be the range of any total function!]

Proof. $\Leftarrow$ follows from our characterisation theorem for c.e. sets.

$\Longrightarrow$ Let $A \neq \emptyset$ be c.e. Suppose is $f$ computable s.t. $A = \mathrm{dom}(f)$.

[This exists again by the characterisation theorem for c.e. sets.]

Fix the number $k \in \mathbb{N}$ and describe what we do to calculate the value $g(k)$ ← $g$ is going to be our total computable fn s.t. $\operatorname{ran}(g) = A$.

Set a flag $:= 0$.

<u>Step $n$</u> $n = \langle i, j \rangle$.

Run the computation $f(w_i)$ for $j$ steps.
If this hasn't halted, go to step $n+1$.
If it has halted, set
$$\text{flag} := \text{flag} + 1.$$
and check whether $\text{flag} = k+1$.
If so, then halt and output $w_i\ [\in A]$.
If not, then go to step $n+1$.

<u>Claim</u> (1) $g$ is a total function.
(2) $\operatorname{ran}(g) = A$.

(1) Since $A \neq \emptyset$, there is some $i \in \mathbb{N}$ s.t. $w_i \in A = \operatorname{dom}(f)$, so $f(w_i)$ halts eventually, say, at $j$.
Then for each $j' \geq j$, the computation of $f(w_i)$ for $j'$ steps has halted.

So at each step

$$n = \langle i, j' \rangle \text{ for } j' \geq j$$

the flag was raised by $1$.

Therefore, for arbitrary $k$, there is some $n$ s.t. in step $n$, the flag was raised to $k+1$. Thus $g(k)\downarrow$.

Since $k$ was arbitrary, this implies ①.

② If $w \in A$ arbitrary, so $w = w_i$.

Then by choice of $f$,

$$w_i \in \mathrm{dom}(f),$$

so the computation $f(w_i)$ halts eventually, say, at $j$.

So in step $n = \langle i, j \rangle$, the flag is raised, say, from $k$ to $k+1$ and therefore $g(k) = w_i = w$.

Thus $\mathrm{ran}(g) = A$.

q.e.d.

<u>Definition</u> A set $A$ is called <u>co-c.e.</u> [complement of computably enumerable]

if $\Sigma^* \setminus A$ is c.e.

<u>Theorem</u> A set $A$ is computable iff it is c.e. and co-c.e.

<u>Proof.</u> $\Longrightarrow$

Suppose $A$ is computable:

$$\chi_A : w \longmapsto \begin{cases} \sigma & \text{if } w \in A \\ \varepsilon & \text{if } w \notin A \end{cases}$$

Then $d_{\uparrow\sigma} \circ \chi_A = \psi_A$, so $A$ is c.e.

In order to get $\psi_{\Sigma^* \setminus A}$:

Run the computation of $\chi_A$ on input $w$.

$\psi_{\Sigma^* \setminus A}$ [ Since $\chi_A$ is total, this will eventually halt and output either $\varepsilon$ or $\sigma$.
If the output is $\sigma$, run into an infinite loop and produce $\uparrow$.
If the output is $\varepsilon$, halt and produce $\sigma$. ]

[REMARK (alternative argument):
The property of being computable is closed under complement:
if $A$ is computable, then
$\Sigma^*\setminus A$ is computable:

$$d_{\sigma\varepsilon} \circ \chi_A = \chi_{\Sigma^*\setminus A}$$

Now use the argument for
"computable $\Longrightarrow$ c.e."
on $\Sigma^*\setminus A$.]

$\Longleftarrow$ Suppose $A$ is c.e. [i.e., $\psi_A$ is computable]
and co-c.e. [i.e., $\psi_{\Sigma^*\setminus A}$ is computable]

We're describing an algorithm for $\chi_A$. Suppose we input $w \in \Sigma^*$.

<u>Step $n$</u>. <u>Case 1</u> $n = 2k$.
Run the computation of $\psi_A(w)$ for $k$ steps.
If it has halted, halt and output $\sigma$.
O/w go to step $n+1$.
<u>Case 2</u> $n = 2k+1$.
Run the computation of $\psi_{\Sigma^*\setminus A}(w)$ for $k$ steps.

If it has halted, halt and output $\varepsilon$.
O/w, go to step $n+1$.

This algorithm halts on every input $w$ and produces the correct output, because

$$\varphi_A(w)\downarrow \iff \varphi_{\Sigma^*\setminus A}(w)\uparrow .$$

q.e.d.

<u>Corollary</u> $\mathbb{K}$, the halting problem, is not co-c.e.

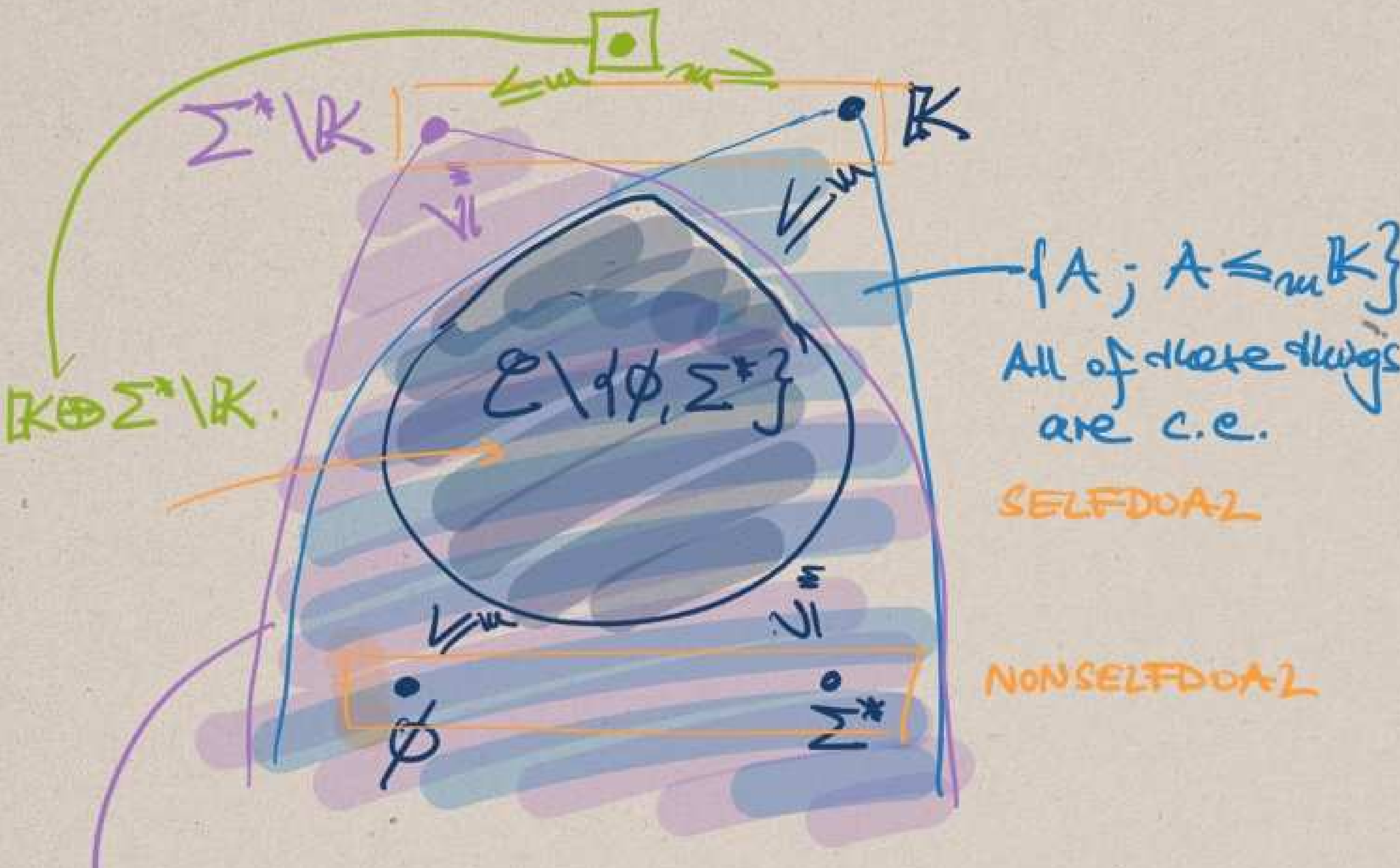
<u>Proof.</u> Clear since it's c.e. and not computable. q.e.d.

So $\Sigma^*\setminus\mathbb{K}$ is a new type of sets in our picture of the degrees on unsolvability:

$$\Sigma^*\setminus\mathbb{K} \not\leq_m \mathbb{K}$$

$$\mathbb{K} \not\leq_m \Sigma^*\setminus\mathbb{K}.$$

[Since this would imply that $\mathbb{K}$ is co-c.e.]




All of these things must be co-c.e.

[• If $A \leq_m B$ and $B$ is c.e., then $A$ is c.e.

• $A \leq_m B \iff \Sigma^*\setminus A \leq_m \Sigma^*\setminus B$.

$\implies$ If $A \leq_m B$ and $B$ is co-c.e.

$\implies \Sigma^*\setminus A \leq_m \Sigma^*\setminus B$ and $\Sigma^*\setminus B$ is c.e.

$\implies \Sigma^*\setminus A$ is c.e.

$\implies$ $A$ is co-c.e. ]

<u>Remark</u> Each of the cones

$$C_{\mathbb{K}} := \{A ; A \leq_m \mathbb{K}\}$$

and $$C_{\Sigma^* \setminus \mathbb{K}} := \{A ; A \leq_m \Sigma^* \setminus \mathbb{K}\}$$

is countable, so their union is countable. Therefore, there must be lots of $A$ s.t. $A \notin C_{\mathbb{K}} \cup C_{\Sigma^* \setminus \mathbb{K}}$.

**Q. Can we find a concrete one?**

<u>Definition</u> $A$ is called <u>selfdual</u> if

$$A \leq_m \Sigma^* \setminus A.$$

[Note that this readily implies $\Sigma^* \setminus A \leq_m A$, so $A \equiv_m \Sigma^* \setminus A$.]

It is called <u>nonselfdual</u> if it is not selfdual.

<u>EXAMPLES</u>. (1) $\emptyset, \Sigma^*, \mathbb{K}, \Sigma^* \setminus \mathbb{K}$ are <u>nonselfdual</u>.

(2) If $A \in \mathcal{E} \setminus \{\emptyset, \Sigma^*\}$, $A$ is <u>selfdual</u>.

<u>Definition</u> Assume that our alphabet has two distinct symbols $\sigma \neq \sigma'$.
Let $A, B \subseteq \Sigma^*$. We define the <u>TURING JOIN OF A AND B</u> by

$$A \oplus B = \{\sigma w \,;\, w \in A\} \cup \{\sigma' w \,;\, w \in B\}.$$

[This is a way of implementing disjoint unions.]

<u>Proposition</u>

$$A \leq_m A \oplus B \qquad (1)$$
$$B \leq_m A \oplus B \qquad (2)$$

<u>Proof.</u> For (1), consider the function

$$f : w \longmapsto \sigma w$$

Then
$$w \in A \iff \sigma w \in \{\sigma w \,;\, w \in A\} \iff \underset{f(w)}{\overset{\shortparallel}{\sigma w}} \in A \oplus B.$$

Similarly, for (2), consider the function

$$g : w \longmapsto \sigma' w.$$

q.e.d.

**Corollary** $\mathbb{K} \oplus \Sigma^* \setminus \mathbb{K}$ is a set that is neither c.e. nor co-c.e. and lies strictly above both $\mathbb{K}$ and $\Sigma^* \setminus \mathbb{K}$ in the degrees of unsolvability.

["Neither c.e. nor co-c.e." follows directly from the Proposition & the fact that $\mathbb{K}$ is not co-c.e.]

**Proposition** $\mathbb{K} \oplus \Sigma^* \setminus \mathbb{K}$ is selfdual.

**Proof.** $C := \mathbb{K} \oplus \Sigma^* \setminus \mathbb{K} = \{\sigma w \,;\, w \in \mathbb{K}\} \cup \{\sigma' w \,;\, w \notin \mathbb{K}\}$

Consider its complement:

$$\Sigma^* \setminus C = \{w \,;\, w \text{ starts with a symbol other than } \sigma \text{ or } \sigma'\}$$

$$\boxed{\begin{array}{l} \cup \{\sigma w \,;\, w \notin \mathbb{K}\} \\ \cup \{\sigma' w \,;\, w \in \mathbb{K}\} \end{array}} \leftarrow \Sigma^* \setminus \mathbb{K} \oplus \mathbb{K}.$$

Let us prove that $C$ and $\Sigma^* \setminus C$ are many-one-equivalent.

Fix some $w_0 \in \mathbb{K}$.

If $w$ starts with any other symbol than $\sigma, \sigma'$, then let

$$f(w) := \sigma w \notin \Sigma^* \setminus \mathbb{K} \oplus \mathbb{K}$$

If $w$ starts with $\sigma$, say $w = \sigma w'$, then $f(w) := \sigma' w'$.

If $w$ starts with $\sigma'$, say, $w = \sigma' w'$, then $f(w) := \sigma w'$.

Check that $f$ is a many-one-reduction from $\mathbb{K} \oplus \Sigma^* \setminus \mathbb{K}$ to $\Sigma^* \setminus \mathbb{K} \oplus \mathbb{K}$.

The other direction follows the same idea.

q.e.d.

Moreover, in general, $A \oplus B$ is the least upper bound of $A$ and $B$ in the degrees of unsolvability.

[As in the following Proposition]

**Proposition** $C \neq \Sigma^*$. If $A \leq_m C$ and $B \leq_m C$, then $A \oplus B \leq_m C$.

**Proof.** $\longrightarrow$ Find $f$ many-one reduction from $A$ to $C$, i.e., $w \in A \iff f(w) \in C$.

$\longrightarrow$ Find $g$ many-one red. from $B$ to $C$, i.e. $w \in B \iff g(w) \in C$.

Find a function $h$ witnessing

$$A \oplus B \leq_m C.$$

Suppose $C \neq \Sigma^*$. Find $v \notin C$.

Given $w$, if $w$ starts with a symbol other than $\sigma, \sigma'$, then $h(w) := v$.

If $w = \sigma w'$, then

$$h(w) := f(w').$$

If $w = \sigma' w'$, then

$$h(w) := g(w').$$

This is a many-one red. of $A \oplus B$ to $C$.

q.e.d.