Recursion Theory

Lecture IX

6 January 2022

two quantifiers

Σ_2 $\quad \boxed{\exists v \forall u} \; (w*v)*u \in R$

Π_2 $\quad \boxed{\forall v \exists u} \; (w*v)*u \in R$

Fin is Σ_2

Inf, Tot are Π_2, moreover Π_2-complete

$$\text{Fin}, \text{Inf}, \text{Tot} \geq_m \mathbb{K} \oplus \Sigma^* \setminus \mathbb{K}$$

Question (*) Are Fin and Inf equivalent?

Are Fin, Inf, Tot strictly more complex than $\mathbb{K} \oplus \Sigma^* \setminus \mathbb{K}$?

It will turn out today that what we did yesterday (Lecture VIII) is enough to answer the Question (*).

Let's look at a simpler class of sets defined by formulas:

Def. $A \subseteq \Sigma^*$ is called

one quantifier

Σ_1 if there is a computable set R s.t.

$$w \in A \iff \boxed{\exists v} \; w*v \in R$$

Π_1 if there is a computable R s.t.

$$w \in A \iff \boxed{\forall v} \; w*v \in R.$$

<u>Observation</u> $\mathbb{K}$ is Σ_1.

$w \in \mathbb{K} \iff \exists u$ the computation $f_w(w)$ has halted after $|u|$ steps.

As before for Σ_2/Π_2:

<u>Observation</u> A is Σ_1 iff $\Sigma^* \setminus A$ is Π_1.

[$w \in A \iff \exists v \; w * v \in R$

$w \notin A \iff \neg \exists v \; w * v \in R$

$\iff \forall v \; \neg \, w * v \in R$

$\iff \forall v \; \underbrace{w * v \notin R}_{\text{the complement of } R \text{ is computable}}$

So: $\Sigma^* \setminus A$ is Π_1.]

<u>Therefore</u> $\Sigma^* \setminus \mathbb{K}$ is Π_1.

<u>Observation</u> If $A \leq_m B$ and B is Σ_1/Π_1, then A is Σ_1/Π_1.

Definition Let $\mathcal{G}$ be a class of subsets of Σ^*, so

$$\mathcal{G} \subseteq \mathcal{P}(\Sigma^*).$$

We say that $\mathcal{G}$ is

- closed under $\leq_m$ — if $\forall A, B$ $A \leq_m B$ and $B \in \mathcal{G}$, then $A \in \mathcal{G}$
- closed under complement — if $A \in \mathcal{G}$, then $\Sigma^* \setminus A \in \mathcal{G}$
- closed under union — if $A, B \in \mathcal{G}$, then $A \cup B \in \mathcal{G}$
- closed under intersections — if $A, B \in \mathcal{G}$, then $A \cap B \in \mathcal{G}$

Examples $\mathcal{C}$ class of computable sets: closed under all four operations

$\mathcal{CE}$ class of c.e. sets: closed under $\leq_m$, not under complement [union & intersection later]

co-$\mathcal{CE}$ class of co-c.e. sets: closed under $\leq_m$, not under complement

$\Sigma_1, \Pi_1, \Sigma_2, \Pi_2$ closed under $\leq_m$

Reformulation of our question:

If Π_2 is not closed under complement, then $\mathrm{Fin} \not\equiv_m \mathrm{Inf}$ and hence

$$\mathrm{Fin}, \mathrm{Inf} >_m \mathbb{K} \oplus \Sigma^* \setminus \mathbb{K}.$$

<u>Theorem</u> (Normal Form Theorem for c.e. sets)

A set is c.e. iff it is Σ_1.

<u>Proof.</u> We remember that we showed that $\mathbb{K}$ is c.e.-complete, i.e., for all $A \in \mathcal{CE}$, $A \leq_m \mathbb{K}$.

<u>Step 1</u> Show that $\mathbb{K}$ is Σ_1-complete.

[Remark. This is precisely the same idea as the proof of "Tot is Π_2-complete", just slightly simpler.]

Fix a Σ_1 set A, i.e.,

$$w \in A \iff \exists v \; w * v \in R.$$

$$g(w,u) := \begin{cases} \varepsilon & \text{if there is a } v \text{ s.t. } w*v \in R \\ \uparrow & \text{o/w} \end{cases}$$

This is computable, so by s-m-n, we get total h s.t.

$$f_{h(w)}(v) = g(w,v)$$

$w \in A$

CASE α

$$f_{h(w)}(v) = \begin{cases} \varepsilon & \text{if } \exists v\ w * v \in R \\ \uparrow & \text{o/w} \end{cases}$$

Case β

$w \notin A$

<u>Case 1.</u> $w \in A$.

This means we're in Case α for every v. So $f_{h(w)}$ is a total function. Thus $h(w) \in \mathbb{K}$.

<u>Case 2</u> $w \notin A$

That means independently of v, we're in Case β. So $f_{h(w)}$ is nowhere defined.

Therefore $h(w) \notin \mathbb{K}$.

This proves that h reduces A to $\mathbb{K}$.

So $\mathbb{K}$ is Σ_1-hard, but we observed earlier that it's Σ_1, so it's Σ_1-complete.

This finishes Step 1.

<u>Step 2.</u> Suppose A is c.e.
Then, by c.e.-completeness of $\mathbb{K}$, we have $A \leq_m \mathbb{K}$.
But $\mathbb{K}$ was Σ_1 and Σ_1 was closed under $\leq_m$.
So A is Σ_1.

<u>Step 3.</u> Suppose A is Σ_1.
Then, by Σ_1-completeness of $\mathbb{K}$, we have $A \leq_m \mathbb{K}$.
But $\mathbb{K}$ was c.e. and $\mathcal{CE}$ is closed under $\leq_m$.
So A is c.e.

q.e.d. (Thm)

<u>Remark.</u> It's easy to argue that $\mathcal{CE}$ and co-$\mathcal{CE}$ are closed under unions and intersections via the computability definitions. Using the Normal Form Thm, this implies that Σ_1, Π_1 are closed under unions & intersections.

We're going to show this directly via the formulas.

Theorem Σ_1 and Π_1 are closed under unions and intersections.

[Only needed for Σ_1 by De Morgan's Law.]

Proof. (1) Unions.

Let A, B be Σ_1

$$w \in A \iff \exists v \; w * v \in R,$$
$$w \in B \iff \exists v \; w * v \in R'.$$

$$\begin{aligned} w \in A \cup B &\iff \exists v \; w*v \in R \lor \exists v \; w*v \in R' \\ &\iff \exists v \, (w*v \in R \lor w*v \in R') \\ &\iff \exists v \, (w*v \in \underbrace{R \cup R'}_{\text{union of computable set, thus computable.}}) \end{aligned}$$

This step tells us why intersections will be more difficult:

We used: $(\exists x \varphi \lor \exists x \psi) \iff \exists x (\varphi \lor \psi)$

But: $(\exists x \varphi \land \exists x \psi) \not\iff \exists x (\varphi \land \psi)$

So, we need to expect some complications for intersections.

(2) Intersections

As before $w \in A \iff \exists v \; w*v \in R$

$w \in B \iff \exists v \; w*v \in R'$

$w \in A \cap B \iff \exists v \; w*v \in R \wedge \exists v \; w*v \in R'$

Remember the functions:

$w \longmapsto w_E$ s.t. $\underline{w_E * w_O = w}$

$w \longmapsto w_O$

$\iff \exists v \left(w*v_E \in R \wedge w*v_O \in R' \right)$

$\Leftarrow$ is clear.

[If v_1 witnesses $w*v_1 \in R$,
v_2 witnesses $w*v_2 \in R'$,
let $v := v_1 * v_2$.
Then $v_E = v_1$, $v_O = v_2$,
so $\exists v \left(w*v_E \in R \wedge w*v_O \in R' \right)$
is true.]

But the set

$$\{ w*v ; \; w*v_E \in R \wedge w*v_O \in R' \}$$

is computable, so the above formula witnesses that $A \cap B$ is Σ_1.

q.e.d.

Theorem Σ_2 and Π_2 are closed under unions and intersections.

Proof. As before, enough to show for Σ_2.

Observe that if A is Σ_2

$$w \in A \iff \exists v \, \forall u \quad (w*v)*u \in R$$

then there is a Π_1 set P s.t.

$$w \in A \iff \exists v \quad w*v \in P.$$

[The Π_1 set is $\{w*v ; \forall u \ (w*v)*u \in R\}$.]

With this in mind, the proof is exactly the same:

(1) Union
$$w \in A \iff \exists v \ w*v \in P$$
$$w \in B \iff \exists v \ w*v \in P'$$
where P, P' are Π_1

$$\begin{aligned} w \in A \cup B &\iff \exists v \ w*v \in P \ \vee \ \exists v \ w*v \in P' \\ &\iff \exists v \ (w*v \in P \ \vee \ w*v \in P') \\ &\iff \exists v \ w*v \in \underline{P \cup P'}. \end{aligned}$$

Since Π_1 is closed under unions, this is Π_1.

(2) Intersections.
As before, $w \in A \iff \exists v \; w * v \in P$
$w \in B \iff \exists v \; w * v \in P'$

$$w \in A \cap B \iff \exists v \; w * v \in P \wedge \exists v \; w * v \in P'.$$

$$\iff \exists v \left(w * v_E \in P \wedge w * v_O \in P' \right)$$

Using closure of Π_1 under intersections (and $\leq_m$), we obtain that this is Π_1.

q.e.d.

<u>Theorem</u> Suppose $\mathcal{G} \subseteq \wp(\Sigma^*)$ is any class of sets with the following properties

① $\mathcal{CE} \subseteq \mathcal{G}$

② $co\mathcal{CE} \subseteq \mathcal{G}$

③ $\mathcal{G}$ is closed under unions and intersections

④ $\mathcal{G}$ is closed under $\leq_m$

⑤ $\mathcal{G}$ is closed under complements.

Then there is no $\mathcal{G}$-complete set.

Corollary Π_2 and Σ_2 are not closed under complements.

[We have just shown that they satisfy all properties ①, ②, ③, and ④.
In Lecture VIII, we proved that Tot is Π_2-complete.
Thus Π_2 cannot be closed under complements (condition ⑤).
By the fact that $\Pi_2 = \text{co-}\Sigma_2$, we get that Σ_2 cannot be closed under complements.]

So, there are some Σ_2 sets that are not Π_2.

Therefore $\mathrm{Fin} \not\equiv_m \mathrm{Inf}$: if $\mathrm{Fin} \equiv_m \mathrm{Inf} \equiv_m \mathrm{Tot}$, then Tot is Σ_2, so by Π_2-completeness of Tot <u>every</u> Π_2 set is Σ_2.

<u>Thus</u>: The abstract theorem solves our question!

$\mathcal{EE} \subseteq \mathcal{G}$ co-$\mathcal{EE} \subseteq \mathcal{G}$

$\mathcal{G}$ is closed under $\le_m$, $\cap$, $\cup$, complem.

Proof of the abstract theorem

Let $\mathcal{G}$ be such a class and towards a contradiction, let's assume that G is $\mathcal{G}$-complete.

Define A by

$$w \in A :\iff w \notin \mathbb{K} \vee (w \in \mathbb{K} \wedge f_w(w) \notin G)$$

1 $\in \mathcal{EE} \subseteq \mathcal{G}$

2 $\in$ co-$\mathcal{EE} \subseteq \mathcal{G}$

5 in co-$\mathcal{G} = \mathcal{G}$

closure under intersections $\longrightarrow \mathcal{G}$ 3 [and 4]

closure under unions $\longrightarrow \mathcal{G}$ 3 [and 4]

We thus showed $A \in \mathcal{G}$.

By $\mathcal{G}$-completeness of G, we have

$$A \le_m G,$$

witnessed by some total computable function h.

Pick program P s.t. $h = f_P$.

$w \in A \iff \underline{w \notin K} \vee (w \in \underline{K} \wedge \overline{f_w(w)} \notin G)$

$A \leq_m G$ by h : $\underline{w \in A \iff h(w) \in G}$

$h = f_P$

Since h is a total function, $P \in K$.

<u>Q.</u> Is $P \in A$?

If $P \in A \implies \underset{\underset{h(P)}{\shortparallel}}{f_P}(P) \notin G$ which contradicts the fact that h is a reduction from A to G.

If $\underline{P \notin A} \implies \underset{\underset{\underline{h(P)}}{\shortparallel}}{f_P}(P) \underline{\in G}$ which contradicts the fact that h is a reduction from A to G.

Therefore : Contradiction!

q.e.d.

<u>Remark.</u> • The definitions $\Sigma_1, \Pi_1, \Sigma_2, \Pi_2$ suggest a general definition for formulas with n quantifiers: $\Sigma_n, \Pi_n \ldots$,

- The proof of closure under unions and intersections translates this closure from Σ_n, Π_n to Σ_{n+1}, Π_{n+1}, and therefore, by induction, all of these classes are closed w.r.t. 3, 4.
- Therefore, our abstract theorem applies to these classes.
- The collection of classes

$$\Sigma_1 \subseteq \Sigma_2 \subseteq \Sigma_3 \subseteq \Sigma_4 \subseteq \ldots$$

is also called the ARITHMETICAL HIERARCHY and we say that it DOESN'T COLLAPSE if all of these inclusions are proper:

$$\Sigma_1 \subsetneq \Sigma_2 \subsetneq \Sigma_3 \subsetneq \Sigma_4 \subsetneq \ldots$$

or, equivalently, that there is no n s.t. Σ_n is closed under complement.

- Therefore, in order to show that the arithmetical hierarchy does not collapse, it is sufficient (by our abstract theorem) to provide a Σ_n-complete set for every n. (Often called Post's Theorem.)

Still to do:

- Show that $\mathbb{K}$ is not an index set.
- Recursion Theorem (Kleene Fixed Pt Thm)
- Other classical properties.

Next lecture:
Wednesday 12 January 2022
$14^{15} - 15^{45}$.