The Open Dihypergraph Dichotomy for Definable Subsets of Generalized Baire Spaces

Dorottya Sziráki

joint work with Philipp Schlicht

Hamburg Set Theory Workshop 2020

The open graph dichotomy for subsets of ${}^\kappa\kappa$

Let κ be an infinite cardinal such that $\kappa^{<\kappa} = \kappa$.

Let $X \subseteq {}^\kappa\kappa$. A graph G on X is an open graph if it is an open subset of $X \times X$.

$\mathrm{OGD}_\kappa(X)$

If G is an open graph on X, then either

- G has a κ-coloring (i.e., X is the union of κ many G-independent sets),
- or G includes a κ-perfect complete subgraph (i.e., there is a continuous injection $f : {}^\kappa 2 \to X$ such that $(f(x), f(y)) \in G$ for all distinct $x, y \in {}^\kappa 2$.)

$\mathrm{OGD}_\kappa(X)$ for definable subsets X of ${}^\kappa\kappa$

Theorem (Feng)

1. $\mathrm{OGD}_\omega(X)$ *holds for all* $\mathbf{\Sigma}^1_1$ *subsets* $X \subseteq {}^\omega\omega$.

$\mathrm{OGD}_\kappa(X)$ for definable subsets X of ${}^\kappa\kappa$

Theorem (Feng)

1. $\mathrm{OGD}_\omega(X)$ *holds for all* $\mathbf{\Sigma}^1_1$ *subsets* $X \subseteq {}^\omega\omega$.
2. *If* λ *is inaccessible, then in any* $\mathrm{Col}(\omega, {<}\lambda)$*-generic extension* $V[G]$, $\mathrm{OGD}_\omega(X)$ *holds for all subsets* $X \subseteq {}^\omega\omega$ *definable from an element of* ${}^\omega\mathrm{Ord}$.

$\mathrm{OGD}_\kappa(X)$ for definable subsets X of ${}^\kappa\kappa$

Theorem (Feng)

1. $\mathrm{OGD}_\omega(X)$ *holds for all* $\mathbf{\Sigma}^1_1$ *subsets* $X \subseteq {}^\omega\omega$.
2. *If* λ *is inaccessible, then in any* $\mathrm{Col}(\omega, {<}\lambda)$*-generic extension* $V[G]$, $\mathrm{OGD}_\omega(X)$ *holds for all subsets* $X \subseteq {}^\omega\omega$ *definable from an element of* ${}^\omega\mathrm{Ord}$.

X is definable from an element of ${}^\omega\mathrm{Ord}$ if

$$X = \{x : \varphi(x, a)\}$$

for some order formula φ with a parameter $a \in {}^\omega\mathrm{Ord}$.

$\mathrm{OGD}_\kappa(X)$ for definable subsets X of ${}^\kappa\kappa$

Theorem (Feng)

1. $\mathrm{OGD}_\omega(X)$ *holds for all* $\mathbf{\Sigma}^1_1$ *subsets* $X \subseteq {}^\omega\omega$.
2. *If* λ *is inaccessible, then in any* $\mathrm{Col}(\omega, {<}\lambda)$*-generic extension* $V[G]$, $\mathrm{OGD}_\omega(X)$ *holds for all subsets* $X \subseteq {}^\omega\omega$ *definable from an element of* ${}^\omega\mathrm{Ord}$.

$\mathrm{OGD}_\kappa(X)$ for definable subsets X of ${}^\kappa\kappa$

Theorem (Feng)

1. *$\mathrm{OGD}_\omega(X)$ holds for all $\mathbf{\Sigma}^1_1$ subsets $X \subseteq {}^\omega\omega$.*
2. *If λ is inaccessible, then in any $\mathrm{Col}(\omega, {<}\lambda)$-generic extension $V[G]$, $\mathrm{OGD}_\omega(X)$ holds for all subsets $X \subseteq {}^\omega\omega$ definable from an element of ${}^\omega\mathrm{Ord}$.*

Suppose κ is an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$.

Theorem (Sz.)

If $\lambda > \kappa$ is inaccessible, then in any $\mathrm{Col}(\kappa, {<}\lambda)$-generic extension $V[G]$, $\mathrm{OGD}_\kappa(X)$ holds for all $\mathbf{\Sigma}^1_1(\kappa)$ subsets $X \subseteq {}^\kappa\kappa$.

$\mathrm{OGD}_\kappa(X)$ for definable subsets X of ${}^\kappa\kappa$

Theorem (Feng)

1. $\mathrm{OGD}_\omega(X)$ *holds for all* $\mathbf{\Sigma}^1_1$ *subsets* $X \subseteq {}^\omega\omega$.
2. *If* λ *is inaccessible, then in any* $\mathrm{Col}(\omega, {<}\lambda)$*-generic extension* $V[G]$, $\mathrm{OGD}_\omega(X)$ *holds for all subsets* $X \subseteq {}^\omega\omega$ *definable from an element of* ${}^\omega\mathrm{Ord}$.

Suppose κ is an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$.

Theorem (Sz.)

If $\lambda > \kappa$ *is inaccessible, then in any* $\mathrm{Col}(\kappa, {<}\lambda)$*-generic extension* $V[G]$, $\mathrm{OGD}_\kappa(X)$ *holds for all* $\mathbf{\Sigma}^1_1(\kappa)$ *subsets* $X \subseteq {}^\kappa\kappa$.

Theorem (Schlicht, Sz.)

In $\mathrm{Col}(\kappa, {<}\lambda)$*-generic extensions, where* $\lambda > \kappa$ *is inaccessible,* $\mathrm{OGD}_\kappa(X)$ *holds for all subsets* $X \subseteq {}^\kappa\kappa$ *definable from an element of* ${}^\kappa\mathrm{Ord}$.

$\mathrm{OGD}_\kappa(X)$ for definable subsets X of ${}^\kappa\kappa$

Theorem (Feng)

1. *$\mathrm{OGD}_\omega(X)$ holds for all $\mathbf{\Sigma}^1_1$ subsets $X \subseteq {}^\omega\omega$.*
2. *If λ is inaccessible, then in any $\mathrm{Col}(\omega, {<}\lambda)$-generic extension $V[G]$, $\mathrm{OGD}_\omega(X)$ holds for all subsets $X \subseteq {}^\omega\omega$ definable from an element of ${}^\omega\mathrm{Ord}$.*

Suppose κ is an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$.

Theorem (Sz.)

If $\lambda > \kappa$ is inaccessible, then in any $\mathrm{Col}(\kappa, {<}\lambda)$-generic extension $V[G]$, $\mathrm{OGD}_\kappa(X)$ holds for all $\mathbf{\Sigma}^1_1(\kappa)$ subsets $X \subseteq {}^\kappa\kappa$.

Theorem (Schlicht, Sz.)

In $\mathrm{Col}(\kappa, {<}\lambda)$-generic extensions, where $\lambda > \kappa$ is inaccessible, $\mathrm{OGD}_\kappa(X)$ holds for all subsets $X \subseteq {}^\kappa\kappa$ definable from an element of ${}^\kappa\mathrm{Ord}$.

These results give the exact consistency strength of these statements.

A higher dimensional version

Introduced in the $\kappa = \omega$ case by R. Carroy, B. Miller and D. Soukup.

A higher dimensional version

Introduced in the $\kappa = \omega$ case by R. Carroy, B. Miller and D. Soukup.

Suppose $\kappa^{<\kappa} = \kappa \geq \omega$. Let $X \subseteq {}^{\kappa}\kappa$ and let $2 \leq \delta \leq \kappa$.

Suppose H is a δ-dimensional dihypergraph on X, i.e., $H \subseteq {}^{\delta}X$ is a set of non-constant sequences.

A higher dimensional version

Introduced in the $\kappa = \omega$ case by R. Carroy, B. Miller and D. Soukup.

Suppose $\kappa^{<\kappa} = \kappa \geq \omega$. Let $X \subseteq {}^{\kappa}\kappa$ and let $2 \leq \delta \leq \kappa$.

Suppose H is a δ-dimensional dihypergraph on X, i.e., $H \subseteq {}^{\delta}X$ is a set of non-constant sequences. H is box-open if it is open in the box topology on ${}^{\delta}X$.

A higher dimensional version

Introduced in the $\kappa = \omega$ case by R. Carroy, B. Miller and D. Soukup.

Suppose $\kappa^{<\kappa} = \kappa \geq \omega$. Let $X \subseteq {}^{\kappa}\kappa$ and let $2 \leq \delta \leq \kappa$.

Suppose H is a δ-dimensional dihypergraph on X, i.e., $H \subseteq {}^{\delta}X$ is a set of non-constant sequences. H is box-open if it is open in the box topology on ${}^{\delta}X$.

$\mathrm{OGD}^{\delta}_{\kappa}(X, H)$

- Either H has a κ-coloring (i.e., X is the union of κ many H-independent sets),

A higher dimensional version

Introduced in the $\kappa = \omega$ case by R. Carroy, B. Miller and D. Soukup.

Suppose $\kappa^{<\kappa} = \kappa \geq \omega$. Let $X \subseteq {}^{\kappa}\kappa$ and let $2 \leq \delta \leq \kappa$.

Suppose H is a δ-dimensional dihypergraph on X, i.e., $H \subseteq {}^{\delta}X$ is a set of non-constant sequences. H is box-open if it is open in the box topology on ${}^{\delta}X$.

$\mathrm{OGD}^{\delta}_{\kappa}(X, H)$

- Either H has a κ-coloring (i.e., X is the union of κ many H-independent sets),
- or there is a continuous map $f : {}^{\kappa}\delta \to X$ which is a homomorphism from $\mathbb{H}_{\delta}$ to H (i.e. $f^{\delta}(\mathbb{H}_{\delta}) \subseteq H$).

A higher dimensional version

Introduced in the $\kappa = \omega$ case by R. Carroy, B. Miller and D. Soukup.

Suppose $\kappa^{<\kappa} = \kappa \geq \omega$. Let $X \subseteq {}^{\kappa}\kappa$ and let $2 \leq \delta \leq \kappa$.

Suppose H is a δ-dimensional dihypergraph on X, i.e., $H \subseteq {}^{\delta}X$ is a set of non-constant sequences. H is box-open if it is open in the box topology on ${}^{\delta}X$.

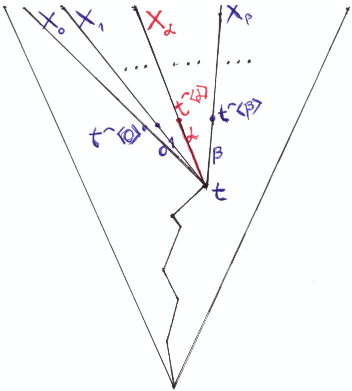
$\mathrm{OGD}^{\delta}_{\kappa}(X, H)$

- Either H has a κ-coloring (i.e., X is the union of κ many H-independent sets),
- or there is a continuous map $f : {}^{\kappa}\delta \to X$ which is a homomorphism from $\mathbb{H}_\delta$ to H (i.e. $f^{\delta}(\mathbb{H}_\delta) \subseteq H$).

$$\mathbb{H}_\delta = \left\{ \overline{x} \in {}^{\delta}({}^{\kappa}\delta) : (\exists t \in {}^{<\kappa}\delta) \; (\forall \alpha < \delta) \; t^\frown\langle\alpha\rangle \subset x_\alpha \right\}.$$

$\mathrm{OGD}^{\delta}_{\kappa}(X)$

$\mathrm{OGD}^{\delta}_{\kappa}(X, H)$ holds for all δ-dimensional box-open dihypergraphs H on X.



$\langle x_0, x_1, \ldots, x_\alpha, \ldots \rangle \in \mathbb{H}_\delta$

$\mathrm{OGD}_\kappa^\delta(X)$

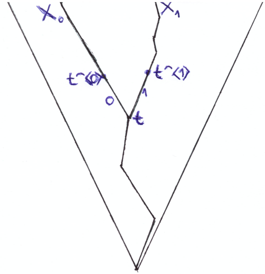
For all δ-dimensional box-open dihypergraphs H on X, either

- H has a κ-coloring, or
- there is a continuous homomorphism $f : {}^\kappa\delta \to X$ from $\mathbb{H}_\delta$ to H.

$\mathrm{OGD}^{\delta}_{\kappa}(X)$

For all δ-dimensional box-open dihypergraphs H on X, either

- H has a κ-coloring, or
- there is a continuous homomorphism $f : {}^{\kappa}\delta \to X$ from $\mathbb{H}_\delta$ to H.



Example

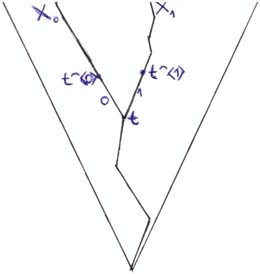
Let $x_0 \neq x_1 \in {}^{\kappa}2$. Let t be the node where they split.

$$\langle x_0, x_1\rangle \in \mathbb{H}_2 \quad \text{iff} \quad x_0(|t|) = 0 \text{ and } x_1(|t|) = 1.$$

$\mathrm{OGD}^{\delta}_{\kappa}(X)$

For all δ-dimensional box-open dihypergraphs H on X, either

- H has a κ-coloring, or
- there is a continuous homomorphism $f : {}^{\kappa}\delta \to X$ from $\mathbb{H}_\delta$ to H.



Example

Let $x_0 \neq x_1 \in {}^{\kappa}2$. Let t be the node where they split.

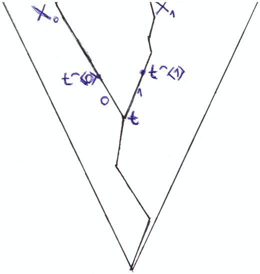
$$\langle x_0, x_1\rangle \in \mathbb{H}_2 \quad \text{iff} \quad x_0(|t|) = 0 \text{ and } x_1(|t|) = 1.$$

The smallest graph (i.e. symmetric relation) containing $\mathbb{H}_2$ is the complete graph $\mathbb{K}_2$ on ${}^{\kappa}2$.

$\mathrm{OGD}_\kappa^\delta(X)$

For all δ-dimensional box-open dihypergraphs H on X, either

- H has a κ-coloring, or
- there is a continuous homomorphism $f : {}^\kappa\delta \to X$ from $\mathbb{H}_\delta$ to H.



Example

Let $x_0 \neq x_1 \in {}^\kappa 2$. Let t be the node where they split.

$$\langle x_0, x_1 \rangle \in \mathbb{H}_2 \quad \text{iff} \quad x_0(|t|) = 0 \text{ and } x_1(|t|) = 1.$$

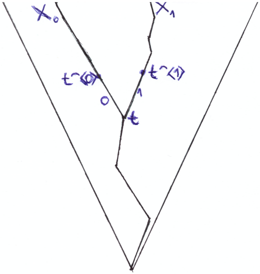
The smallest graph (i.e. symmetric relation) containing $\mathbb{H}_2$ is the complete graph $\mathbb{K}_2$ on ${}^\kappa 2$.

G is a graph on X. Let $f : {}^\kappa 2 \to X$ be continuous homomorphism from $\mathbb{H}_2$ to G.

$\mathrm{OGD}^{\delta}_{\kappa}(X)$

For all δ-dimensional box-open dihypergraphs H on X, either

- H has a κ-coloring, or
- there is a continuous homomorphism $f : {}^{\kappa}\delta \to X$ from $\mathbb{H}_\delta$ to H.



Example

Let $x_0 \neq x_1 \in {}^{\kappa}2$. Let t be the node where they split.

$$\langle x_0, x_1\rangle \in \mathbb{H}_2 \quad \text{iff} \quad x_0(|t|) = 0 \text{ and } x_1(|t|) = 1.$$

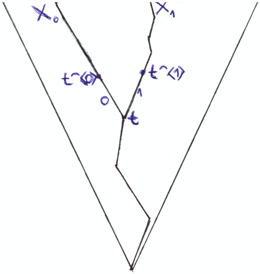
The smallest graph (i.e. symmetric relation) containing $\mathbb{H}_2$ is the complete graph $\mathbb{K}_2$ on ${}^{\kappa}2$.

G is a graph on X. Let $f : {}^{\kappa}2 \to X$ be continuous homomorphism from $\mathbb{H}_2$ to G. Since G is symmetric, f is a homomorphism from $\mathbb{K}_2$ to G.

$\mathrm{OGD}^{\delta}_{\kappa}(X)$

For all δ-dimensional box-open dihypergraphs H on X, either

- H has a κ-coloring, or
- there is a continuous homomorphism $f : {}^{\kappa}\delta \to X$ from $\mathbb{H}_\delta$ to H.



Example

Let $x_0 \neq x_1 \in {}^{\kappa}2$. Let t be the node where they split.

$$\langle x_0, x_1\rangle \in \mathbb{H}_2 \quad \text{iff} \quad x_0(|t|) = 0 \text{ and } x_1(|t|) = 1.$$

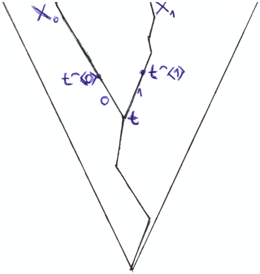
The smallest graph (i.e. symmetric relation) containing $\mathbb{H}_2$ is the complete graph $\mathbb{K}_2$ on ${}^{\kappa}2$.

G is a graph on X. Let $f : {}^{\kappa}2 \to X$ be continuous homomorphism from $\mathbb{H}_2$ to G. Since G is symmetric, f is a homomorphism from $\mathbb{K}_2$ to G. Thus, G has a κ-perfect complete subgraph.

$\mathrm{OGD}^\delta_\kappa(X)$

For all δ-dimensional box-open dihypergraphs H on X, either

- H has a κ-coloring, or
- there is a continuous homomorphism $f : {}^\kappa\delta \to X$ from $\mathbb{H}_\delta$ to H.



Example

Let $x_0 \neq x_1 \in {}^\kappa 2$. Let t be the node where they split.

$$\langle x_0, x_1\rangle \in \mathbb{H}_2 \quad \text{iff} \quad x_0(|t|) = 0 \text{ and } x_1(|t|) = 1.$$

The smallest graph (i.e. symmetric relation) containing $\mathbb{H}_2$ is the complete graph $\mathbb{K}_2$ on ${}^\kappa 2$.

G is a graph on X. Let $f : {}^\kappa 2 \to X$ be continuous homomorphism from $\mathbb{H}_2$ to G. Since G is symmetric, f is a homomorphism from $\mathbb{K}_2$ to G. Thus, G has a κ-perfect complete subgraph.

$\mathrm{OGD}^2_\kappa(X)$ implies the open graph dichotomy $\mathrm{OGD}_\kappa(X)$.

Applications of $\mathrm{OGD}^{\omega}_{\omega}(X)$

Theorem (R. Carroy, B. Miller, D. Soukup)

$\mathrm{OGD}^{\omega}_{\omega}(X)$ *holds for all* $\mathbf{\Sigma}^1_1$ *subsets* X *of* ${}^{\omega}\omega$

Applications of $\mathrm{OGD}^{\omega}_{\omega}(X)$

Theorem (R. Carroy, B. Miller, D. Soukup)

$\mathrm{OGD}^{\omega}_{\omega}(X)$ *holds for all* $\mathbf{\Sigma}^1_1$ *subsets* X *of* ${}^{\omega}\omega$ *(and more generally, for all analytic Hausdorff spaces).*

Applications of $\mathrm{OGD}^{\omega}_{\omega}(X)$

Theorem (R. Carroy, B. Miller, D. Soukup)

$\mathrm{OGD}^{\omega}_{\omega}(X)$ holds for all $\mathbf{\Sigma}^1_1$ subsets X of ${}^{\omega}\omega$ (and more generally, for all analytic Hausdorff spaces).

Theorem (R. Carroy, B. Miller, D. Soukup)

Suppose X is a separable metric space such that $\mathrm{OGD}^{\omega}_{\omega}(X)$ holds.

- *X satisfies the Hurewicz dichotomy (characterizes when X is contained in a K_σ subset of ${}^{\omega}\omega$).*

Applications of $\mathrm{OGD}^{\omega}_{\omega}(X)$

Theorem (R. Carroy, B. Miller, D. Soukup)

$\mathrm{OGD}^{\omega}_{\omega}(X)$ *holds for all* $\mathbf{\Sigma}^1_1$ *subsets* X *of* ${}^{\omega}\omega$ *(and more generally, for all analytic Hausdorff spaces).*

Theorem (R. Carroy, B. Miller, D. Soukup)

Suppose X *is a separable metric space such that* $\mathrm{OGD}^{\omega}_{\omega}(X)$ *holds.*

- X *satisfies the Hurewicz dichotomy (characterizes when* X *is contained in a* K_σ *subset of* ${}^{\omega}\omega$*).*
- *The Jayne-Rogers theorem holds for* X *(characterizes when a given function from* X *to a separable metric space is* $\mathbf{\Delta}^0_2$*-measurable).*

Applications of $\mathrm{OGD}^\omega_\omega(X)$

Theorem (R. Carroy, B. Miller, D. Soukup)

$\mathrm{OGD}^\omega_\omega(X)$ *holds for all* $\mathbf{\Sigma}^1_1$ *subsets* X *of* ${}^\omega\omega$ *(and more generally, for all analytic Hausdorff spaces).*

Theorem (R. Carroy, B. Miller, D. Soukup)

Suppose X *is a separable metric space such that* $\mathrm{OGD}^\omega_\omega(X)$ *holds.*

- X *satisfies the Hurewicz dichotomy (characterizes when* X *is contained in a* K_σ *subset of* ${}^\omega\omega$*).*
- *The Jayne-Rogers theorem holds for* X *(characterizes when a given function from* X *to a separable metric space is* $\mathbf{\Delta}^0_2$*-measurable).*
- *A theorem of Lecomte and Zeleny holds for* X*, which characterizes when a graph on* X *has* $\mathbf{\Delta}^0_2$*-measurable* $\aleph_0$*-coloring.*

Applications of $\mathrm{OGD}^{\omega}_{\omega}(X)$

Theorem (R. Carroy, B. Miller, D. Soukup)

$\mathrm{OGD}^{\omega}_{\omega}(X)$ holds for all $\mathbf{\Sigma}^1_1$ subsets X of ${}^{\omega}\omega$ (and more generally, for all analytic Hausdorff spaces).

Theorem (R. Carroy, B. Miller, D. Soukup)

Suppose X is a separable metric space such that $\mathrm{OGD}^{\omega}_{\omega}(X)$ holds.

- *X satisfies the Hurewicz dichotomy (characterizes when X is contained in a K_σ subset of ${}^{\omega}\omega$).*
- *The Jayne-Rogers theorem holds for X (characterizes when a given function from X to a separable metric space is $\mathbf{\Delta}^0_2$-measurable).*
- *A theorem of Lecomte and Zeleny holds for X, which characterizes when a graph on X has $\mathbf{\Delta}^0_2$-measurable $\aleph_0$-coloring.*
- *Several other applications ...*

$\mathrm{OGD}^{\delta}_{\kappa}(X)$ for definable subsets of ${}^{\kappa}\kappa$

Theorem (Schlicht, Sz.)

Suppose $\kappa^{<\kappa} = \kappa \geq \omega$. If $\lambda > \kappa$ is inaccessible, then in any $\mathrm{Col}(\kappa, <\lambda)$-generic extension $V[G]$, the following hold for all subsets $X \subseteq {}^{\kappa}\kappa$ which are definable from an element of ${}^{\kappa}\mathrm{Ord}$:

1. $\mathrm{OGD}^{\delta}_{\kappa}(X)$, *where* $2 \leq \delta < \kappa$.

$\mathrm{OGD}^{\delta}_{\kappa}(X)$ for definable subsets of ${}^{\kappa}\kappa$

Theorem (Schlicht, Sz.)

Suppose $\kappa^{<\kappa} = \kappa \geq \omega$. If $\lambda > \kappa$ is inaccessible, then in any $\mathrm{Col}(\kappa, {<}\lambda)$-generic extension $V[G]$, the following hold for all subsets $X \subseteq {}^{\kappa}\kappa$ which are definable from an element of ${}^{\kappa}\mathrm{Ord}$:

1. $\mathrm{OGD}^{\delta}_{\kappa}(X)$, *where* $2 \leq \delta < \kappa$.
2. $\mathrm{OGD}^{\kappa}_{\kappa}(X, H)$ *for all κ-dimensional box-open dihypergraphs H on X which are definable from an element of ${}^{\kappa}\mathrm{Ord}$.*

$\mathrm{OGD}^{\delta}_{\kappa}(X)$ for definable subsets of ${}^{\kappa}\kappa$

Theorem (Schlicht, Sz.)

Suppose $\kappa^{<\kappa} = \kappa \geq \omega$. If $\lambda > \kappa$ is inaccessible, then in any $\mathrm{Col}(\kappa, {<}\lambda)$-generic extension $V[G]$, the following hold for all subsets $X \subseteq {}^{\kappa}\kappa$ which are definable from an element of ${}^{\kappa}\mathrm{Ord}$:

1. $\mathrm{OGD}^{\delta}_{\kappa}(X)$, *where $2 \leq \delta < \kappa$.*
2. $\mathrm{OGD}^{\kappa}_{\kappa}(X, H)$ *for all κ-dimensional box-open dihypergraphs H on X which are definable from an element of ${}^{\kappa}\mathrm{Ord}$.*

This theorem gives the exact consistency strength of these statements.

Sketch of the proof

$\kappa^{<\kappa} = \kappa \geq \omega$. Let $\lambda > \kappa$ be inaccessible, let G be $\mathrm{Col}(\kappa, {<}\lambda)$-generic over V.

Sketch of the proof

$\kappa^{<\kappa} = \kappa \geq \omega$. Let $\lambda > \kappa$ be inaccessible, let G be $\mathrm{Col}(\kappa, {<}\lambda)$-generic over V.
For all $\alpha \leq \lambda$, let $\mathbb{P}_\alpha = \mathrm{Col}(\kappa, {<}\alpha)$ and $G_\alpha = G \cap \mathbb{P}_\alpha$.

Sketch of the proof

$\kappa^{<\kappa} = \kappa \geq \omega$. Let $\lambda > \kappa$ be inaccessible, let G be $\mathrm{Col}(\kappa, {<}\lambda)$-generic over V.

For all $\alpha \leq \lambda$, let $\mathbb{P}_\alpha = \mathrm{Col}(\kappa, {<}\alpha)$ and $G_\alpha = G \cap \mathbb{P}_\alpha$.

In $V[G]$, assume:

- $X \subseteq {}^\kappa\kappa$ is defined by a formula φ_X with a parameter $a_X \in {}^\kappa\mathrm{Ord}$. That is,
$$X = \{x \in ({}^\kappa\kappa)^{V[G]} : V[G] \models \varphi_X(x, a_X)\}.$$

Sketch of the proof

$\kappa^{<\kappa} = \kappa \geq \omega$. Let $\lambda > \kappa$ be inaccessible, let G be $\mathrm{Col}(\kappa, {<}\lambda)$-generic over V.

For all $\alpha \leq \lambda$, let $\mathbb{P}_\alpha = \mathrm{Col}(\kappa, {<}\alpha)$ and $G_\alpha = G \cap \mathbb{P}_\alpha$.

In $V[G]$, assume:

- $X \subseteq {}^\kappa\kappa$ is defined by a formula φ_X with a parameter $a_X \in {}^\kappa\mathrm{Ord}$. That is,
$$X = \{x \in ({}^\kappa\kappa)^{V[G]} : V[G] \models \varphi_X(x, a_X)\}.$$
- R is a δ-dimensional box-open dihypergraph on X which has no κ-coloring.

Sketch of the proof

$\kappa^{<\kappa} = \kappa \geq \omega$. Let $\lambda > \kappa$ be inaccessible, let G be $\mathrm{Col}(\kappa, {<}\lambda)$-generic over V.

For all $\alpha \leq \lambda$, let $\mathbb{P}_\alpha = \mathrm{Col}(\kappa, {<}\alpha)$ and $G_\alpha = G \cap \mathbb{P}_\alpha$.

In $V[G]$, assume:

- $X \subseteq {}^\kappa\kappa$ is defined by a formula φ_X with a parameter $a_X \in {}^\kappa\mathrm{Ord}$. That is,
$$X = \{x \in ({}^\kappa\kappa)^{V[G]} : V[G] \models \varphi_X(x, a_X)\}.$$
- R is a δ-dimensional box-open dihypergraph on X which has no κ-coloring.
- R is defined by a formula ψ_R with a parameter $b_R \in {}^\kappa\mathrm{Ord}$. That is,
$$R = \{\overline{x} \in ({}^\delta({}^\kappa\kappa))^{V[G]} : V[G] \models \psi_R(\overline{x}, b_R)\}).$$

Sketch of the proof

$\kappa^{<\kappa} = \kappa \geq \omega$. Let $\lambda > \kappa$ be inaccessible, let G be $\mathrm{Col}(\kappa, {<}\lambda)$-generic over V.

For all $\alpha \leq \lambda$, let $\mathbb{P}_\alpha = \mathrm{Col}(\kappa, {<}\alpha)$ and $G_\alpha = G \cap \mathbb{P}_\alpha$.

In $V[G]$, assume:

- $X \subseteq {}^\kappa\kappa$ is defined by a formula φ_X with a parameter $a_X \in {}^\kappa\mathrm{Ord}$. That is,
$$X = \{x \in ({}^\kappa\kappa)^{V[G]} : V[G] \models \varphi_X(x, a_X)\}.$$
- R is a δ-dimensional box-open dihypergraph on X which has no κ-coloring.
- R is defined by a formula ψ_R with a parameter $b_R \in {}^\kappa\mathrm{Ord}$. That is,
$$R = \{\overline{x} \in ({}^\delta({}^\kappa\kappa))^{V[G]} : V[G] \models \psi_R(\overline{x}, b_R)\}).$$

  (When $\delta < \kappa$, this can be assumed whenever R is box-open.)

Sketch of the proof

$\kappa^{<\kappa} = \kappa \geq \omega$. Let $\lambda > \kappa$ be inaccessible, let G be $\mathrm{Col}(\kappa, {<}\lambda)$-generic over V.

For all $\alpha \leq \lambda$, let $\mathbb{P}_\alpha = \mathrm{Col}(\kappa, {<}\alpha)$ and $G_\alpha = G \cap \mathbb{P}_\alpha$.

In $V[G]$, assume:

- $X \subseteq {}^\kappa\kappa$ is defined by a formula φ_X with a parameter $a_X \in {}^\kappa\mathrm{Ord}$. That is,
$$X = \{x \in ({}^\kappa\kappa)^{V[G]} : V[G] \models \varphi_X(x, a_X)\}.$$
- R is a δ-dimensional box-open dihypergraph on X which has no κ-coloring.
- R is defined by a formula ψ_R with a parameter $b_R \in {}^\kappa\mathrm{Ord}$. That is,
$$R = \{\overline{x} \in ({}^\delta({}^\kappa\kappa))^{V[G]} : V[G] \models \psi_R(\overline{x}, b_R)\}).$$

  (When $\delta < \kappa$, this can be assumed whenever R is box-open.)
- We can also assume that $a_X, b_R \in V$.

Sketch of the proof

$\kappa^{<\kappa} = \kappa \geq \omega$. Let $\lambda > \kappa$ be inaccessible, let G be $\mathrm{Col}(\kappa, {<}\lambda)$-generic over V.

For all $\alpha \leq \lambda$, let $\mathbb{P}_\alpha = \mathrm{Col}(\kappa, {<}\alpha)$ and $G_\alpha = G \cap \mathbb{P}_\alpha$.

In $V[G]$, assume:

- $X \subseteq {}^\kappa\kappa$ is defined by a formula φ_X with a parameter $a_X \in {}^\kappa\mathrm{Ord}$. That is,
$$X = \{x \in ({}^\kappa\kappa)^{V[G]} : V[G] \models \varphi_X(x, a_X)\}.$$
- R is a δ-dimensional box-open dihypergraph on X which has no κ-coloring.
- R is defined by a formula ψ_R with a parameter $b_R \in {}^\kappa\mathrm{Ord}$. That is,
$$R = \{\overline{x} \in ({}^\delta({}^\kappa\kappa))^{V[G]} : V[G] \models \psi_R(\overline{x}, b_R)\}).$$
(When $\delta < \kappa$, this can be assumed whenever R is box-open.)
- We can also assume that $a_X, b_R \in V$.

$$X - \bigcup\{[T] : T \in V \text{ is a subtree of } {}^{<\kappa}\kappa,\ [T] \text{ is } R\text{-independent}\}.$$

Sketch of the proof

$\kappa^{<\kappa} = \kappa \geq \omega$. Let $\lambda > \kappa$ be inaccessible, let G be $\mathrm{Col}(\kappa, {<}\lambda)$-generic over V.

For all $\alpha \leq \lambda$, let $\mathbb{P}_\alpha = \mathrm{Col}(\kappa, {<}\alpha)$ and $G_\alpha = G \cap \mathbb{P}_\alpha$.

In $V[G]$, assume:

- $X \subseteq {}^\kappa\kappa$ is defined by a formula φ_X with a parameter $a_X \in {}^\kappa\mathrm{Ord}$. That is,
$$X = \{x \in ({}^\kappa\kappa)^{V[G]} : V[G] \models \varphi_X(x, a_X)\}.$$
- R is a δ-dimensional box-open dihypergraph on X which has no κ-coloring.
- R is defined by a formula ψ_R with a parameter $b_R \in {}^\kappa\mathrm{Ord}$. That is,
$$R = \{\overline{x} \in ({}^\delta({}^\kappa\kappa))^{V[G]} : V[G] \models \psi_R(\overline{x}, b_R)\}).$$
(When $\delta < \kappa$, this can be assumed whenever R is box-open.)
- We can also assume that $a_X, b_R \in V$.

Let $x \in X - \bigcup\{[T] : T \in V$ is a subtree of ${}^{<\kappa}\kappa$, $[T]$ is R-independent$\}$.

Sketch of the proof

$\kappa^{<\kappa} = \kappa \geq \omega$. Let $\lambda > \kappa$ be inaccessible, let G be $\mathrm{Col}(\kappa, {<}\lambda)$-generic over V.

For all $\alpha \leq \lambda$, let $\mathbb{P}_\alpha = \mathrm{Col}(\kappa, {<}\alpha)$ and $G_\alpha = G \cap \mathbb{P}_\alpha$.

In $V[G]$, assume:

- $X \subseteq {}^\kappa\kappa$ is defined by a formula φ_X with a parameter $a_X \in {}^\kappa\mathrm{Ord}$. That is,
$$X = \{x \in ({}^\kappa\kappa)^{V[G]} : V[G] \models \varphi_X(x, a_X)\}.$$
- R is a δ-dimensional box-open dihypergraph on X which has no κ-coloring.
- R is defined by a formula ψ_R with a parameter $b_R \in {}^\kappa\mathrm{Ord}$. That is,
$$R = \{\overline{x} \in ({}^\delta({}^\kappa\kappa))^{V[G]} : V[G] \models \psi_R(\overline{x}, b_R)\}).$$
(When $\delta < \kappa$, this can be assumed whenever R is box-open.)
- We can also assume that $a_X, b_R \in V$.

Let $x \in X - \bigcup\{[T] : T \in V$ is a subtree of ${}^{<\kappa}\kappa$, $[T]$ is R-independent$\}$.

Then $x \in V[G_\alpha]$ for some $\alpha < \lambda$. Let $\dot{x}$ be a $\mathbb{P}_\alpha$-name for x.

Sketch of the proof (the $\kappa = \omega$ case)

For $\kappa = \omega$, the theorem can be proved using an argument similar to Feng's proof, and to an argument of Solovay's.

Sketch of the proof (the $\kappa = \omega$ case)

For $\kappa = \omega$, the theorem can be proved using an argument similar to Feng's proof, and to an argument of Solovay's. These arguments rely on the following lemma.

Lemma 1 (Solovay)

For all countable sequences y of ordinals in $V[G]$, $V[G]$ is a $\mathbb{P}_\lambda$-generic extension of $V[y]$.

- This lemma fails when $\kappa > \omega$ (Schlicht).

Sketch of the proof (the $\kappa = \omega$ case)

For $\kappa = \omega$, the theorem can be proved using an argument similar to Feng's proof, and to an argument of Solovay's. These arguments rely on the following lemma.

Lemma 1 (Solovay)

For all countable sequences y of ordinals in $V[G]$, $V[G]$ is a $\mathbb{P}_\lambda$-generic extension of $V[y]$.

- This lemma fails when $\kappa > \omega$ (Schlicht).

We construct a $\subsetneq$-preserving map $e : {}^{<\kappa}\delta \to \mathbb{P}_\alpha$ such that for all $y \in {}^{\kappa}\delta$,

$$g_y = \{q \in \mathbb{P}_\alpha : q \geq e(t) \text{ for some } t \subsetneq y\} \text{ is a } \mathbb{P}_\alpha\text{-generic filter.}$$

Sketch of the proof (the $\kappa = \omega$ case)

For $\kappa = \omega$, the theorem can be proved using an argument similar to Feng's proof, and to an argument of Solovay's. These arguments rely on the following lemma.

Lemma 1 (Solovay)

For all countable sequences y of ordinals in $V[G]$, $V[G]$ is a $\mathbb{P}_\lambda$-generic extension of $V[y]$.

- This lemma fails when $\kappa > \omega$ (Schlicht).

We construct a $\subsetneq$-preserving map $e : {}^{<\kappa}\delta \to \mathbb{P}_\alpha$ such that for all $y \in {}^{\kappa}\delta$,

$$g_y = \{q \in \mathbb{P}_\alpha : \ q \geq e(t) \text{ for some } t \subsetneq y\} \text{ is a } \mathbb{P}_\alpha\text{-generic filter.}$$

By the next lemma, e can be defined in such a way that $\dot{x}^{g_y} \in X$ for all $y \in {}^{\kappa}\delta$, and the (continuous) map

$$f : {}^{\kappa}\delta \to X; \ y \mapsto \dot{x}^{g_y}$$

is a homomorphism from $\mathbb{H}_\delta$ to H.

Sketch of the proof (the $\kappa = \omega$ case)

For any forcing $\mathbb{Q}$, any $q \in \mathbb{Q}$ and any $\mathbb{Q}$-name σ, define

$$T_{\mathbb{Q}}^{\sigma,q} = \{t \in {}^{<\kappa}\kappa : (\exists r \leq q)\, r \Vdash_{\mathbb{Q}}^{V} t \subseteq \sigma\},$$

the tree of possible values for σ below q.

Sketch of the proof (the $\kappa = \omega$ case)

For any forcing $\mathbb{Q}$, any $q \in \mathbb{Q}$ and any $\mathbb{Q}$-name σ, define

$$T^{\sigma,q}_{\mathbb{Q}} = \{t \in {}^{<\kappa}\kappa : (\exists r \leq q)\, r \Vdash^{V}_{\mathbb{Q}} t \subseteq \sigma\},$$

the tree of possible values for σ below q.

$\dot{x}^{G_\alpha} \in X$; if $T \in V$ is a subtree of ${}^{<\kappa}\kappa$ and $[T]$ is R-independent, then $\dot{x}^{G_\alpha} \notin [T]$.

Lemma 2

There exists $p \in \mathbb{P}_\alpha$ such that the following hold.

1. $p \Vdash^{V}_{\mathbb{P}_\alpha}$ "$\varphi_X(\dot{x}, a_X)$ holds in every further $\mathbb{P}_\lambda$-generic extension of $V[\dot{x}]$."
2. For all $r \in \mathbb{P}_\alpha$ below p, there exists (in $V[G]$) a sequence $\langle t_i \in T^{\dot{x},r}_{\mathbb{P}_\alpha} : i < \delta\rangle$ such that (in $V[G]$)
$$\prod_{i<\delta} N_{t_i} \cap X \subseteq R.$$

Sketch of the proof (the $\kappa > \omega$ case)

Assume $\kappa = \kappa^{<\kappa}$ is uncountable.

Lemma 3

There exist $\gamma < \lambda$ and an $\mathrm{Add}(\kappa, 1)$-name $\tau \in V[G_\gamma]$ which satisfy a strong version of Lemma 2:

Sketch of the proof (the $\kappa > \omega$ case)

Assume $\kappa = \kappa^{<\kappa}$ is uncountable.

Lemma 3

There exist $\gamma < \lambda$ and an $\mathrm{Add}(\kappa, 1)$-name $\tau \in V[G_\gamma]$ which satisfy a strong version of Lemma 2:

1. $\Vdash^{V[G_\gamma]}_{\mathrm{Add}(\kappa,1)}$ "$\varphi_X(\tau, a_X)$ holds in every further $\mathbb{P}_\lambda$-generic extension."

Sketch of the proof (the $\kappa > \omega$ case)

Assume $\kappa = \kappa^{<\kappa}$ is uncountable.

Lemma 3

There exist $\gamma < \lambda$ and an $\mathrm{Add}(\kappa, 1)$-name $\tau \in V[G_\gamma]$ which satisfy a strong version of Lemma 2:

1. $\Vdash^{V[G_\gamma]}_{\mathrm{Add}(\kappa,1)}$ "$\varphi_X(\tau, a_X)$ holds in every further $\mathbb{P}_\lambda$-generic extension."
2. For all $r \in \mathrm{Add}(\kappa, 1)$, there exists a sequence

$$\bar{t}(r) = \langle t_i(r) \in T^{\tau,r}_{\mathrm{Add}(\kappa,1)} : i < \delta \rangle \in V[G_\gamma]$$

such that in $V[G]$,

$$\prod_{i<\delta} N_{t_i(r)} \cap X \subseteq R.$$

Sketch of the proof (the $\kappa > \omega$ case)

Let $\mathbb{Q}$ consist of those partial maps p from ${}^{<\kappa}\delta$ to ${}^{<\kappa}\kappa$ such that

1. $\mathrm{dom}(p)$ is a subtree of ${}^{<\kappa}\delta$ of size $< \kappa$.
2. For all $u, v \in \mathrm{dom}(p)$, $u \subsetneq v$ implies $p(u) \subsetneq p(v)$.

Sketch of the proof (the $\kappa > \omega$ case)

Let $\mathbb{Q}$ consist of those partial maps p from ${}^{<\kappa}\delta$ to ${}^{<\kappa}\kappa$ such that

1. $\mathrm{dom}(p)$ is a subtree of ${}^{<\kappa}\delta$ of size $< \kappa$.
2. For all $u, v \in \mathrm{dom}(p)$, $u \subsetneq v$ implies $p(u) \subsetneq p(v)$.
3. If $i < \delta$ and $u^\frown\langle i\rangle \in \mathrm{dom}(p)$, then $p\big(u^\frown\langle i\rangle\big) \Vdash_{\mathrm{Add}(\kappa,1)} t_i\big(p(u)\big) \subseteq \tau$.

Sketch of the proof (the $\kappa > \omega$ case)

Let $\mathbb{Q}$ consist of those partial maps p from ${}^{<\kappa}\delta$ to ${}^{<\kappa}\kappa$ such that

1. $\mathrm{dom}(p)$ is a subtree of ${}^{<\kappa}\delta$ of size $< \kappa$.
2. For all $u, v \in \mathrm{dom}(p)$, $u \subsetneq v$ implies $p(u) \subsetneq p(v)$.
3. If $i < \delta$ and $u^\frown\langle i\rangle \in \mathrm{dom}(p)$, then $p\big(u^\frown\langle i\rangle\big) \Vdash_{\mathrm{Add}(\kappa,1)} t_i\big(p(u)\big) \subseteq \tau$.

We let $p \leq_{\mathbb{Q}} q$ if and only if $\mathrm{dom}(p) \supseteq \mathrm{dom}(q)$, and

- $p(u) = q(u)$ for every non-terminal node $u \in \mathrm{dom}(q)$, and
- $p(u) \supseteq q(u)$ for every terminal node u of $\mathrm{dom}(q)$.

Sketch of the proof (the $\kappa > \omega$ case)

Let $\mathbb{Q}$ consist of those partial maps p from ${}^{<\kappa}\delta$ to ${}^{<\kappa}\kappa$ such that

1. $\operatorname{dom}(p)$ is a subtree of ${}^{<\kappa}\delta$ of size $< \kappa$.
2. For all $u, v \in \operatorname{dom}(p)$, $u \subsetneq v$ implies $p(u) \subsetneq p(v)$.
3. If $i < \delta$ and $u^\frown\langle i\rangle \in \operatorname{dom}(p)$, then $p\big(u^\frown\langle i\rangle\big) \Vdash_{\mathrm{Add}(\kappa,1)} t_i\big(p(u)\big) \subseteq \tau$.

We let $p \leq_{\mathbb{Q}} q$ if and only if $\operatorname{dom}(p) \supseteq \operatorname{dom}(q)$, and

- $p(u) = q(u)$ for every non-terminal node $u \in \operatorname{dom}(q)$, and
- $p(u) \supseteq q(u)$ for every terminal node u of $\operatorname{dom}(q)$.

A $\mathbb{Q}$-generic filter H adds a $\subsetneq$-preserving map $e_H : {}^{<\kappa}\delta \to {}^{<\kappa}\kappa$;

$$e_H(u) = \bigcup\{p(u) : p \in H\}.$$

Sketch of the proof (the $\kappa > \omega$ case)

Let $\mathbb{Q}$ consist of those partial maps p from ${}^{<\kappa}\delta$ to ${}^{<\kappa}\kappa$ such that

1. $\operatorname{dom}(p)$ is a subtree of ${}^{<\kappa}\delta$ of size $< \kappa$.
2. For all $u, v \in \operatorname{dom}(p)$, $u \subsetneq v$ implies $p(u) \subsetneq p(v)$.
3. If $i < \delta$ and $u^\frown\langle i\rangle \in \operatorname{dom}(p)$, then $p\big(u^\frown\langle i\rangle\big) \Vdash_{\mathrm{Add}(\kappa,1)} t_i\big(p(u)\big) \subseteq \tau$.

We let $p \leq_{\mathbb{Q}} q$ if and only if $\operatorname{dom}(p) \supseteq \operatorname{dom}(q)$, and

- $p(u) = q(u)$ for every non-terminal node $u \in \operatorname{dom}(q)$, and
- $p(u) \supseteq q(u)$ for every terminal node u of $\operatorname{dom}(q)$.

A $\mathbb{Q}$-generic filter H adds a $\subsetneq$-preserving map $e_H : {}^{<\kappa}\delta \to {}^{<\kappa}\kappa$;

$$e_H(u) = \bigcup\{p(u) : p \in H\}.$$

$\mathbb{Q}$ is equivalent to $\mathrm{Add}(\kappa, 1)$, since it is $<\kappa$-closed, nonatomic, and of size κ.

Sketch of the proof (the $\kappa > \omega$ case)

Let $K \times H'$ be $\mathbb{P}_\lambda \times \mathrm{Add}(\kappa, 1)$-generic over $V[G_\gamma]$ with $V[G] = V[G_\gamma \times K \times H']$.

Sketch of the proof (the $\kappa > \omega$ case)

Let $K \times H'$ be $\mathbb{P}_\lambda \times \mathrm{Add}(\kappa, 1)$-generic over $V[G_\gamma]$ with $V[G] = V[G_\gamma \times K \times H']$.

Replace H' with a $\mathbb{Q}$-generic H over $V[G_\gamma \times K]$ such that $V[G] = V[G_\gamma \times K \times H]$.

Sketch of the proof (the $\kappa > \omega$ case)

Let $K \times H'$ be $\mathbb{P}_\lambda \times \mathrm{Add}(\kappa, 1)$-generic over $V[G_\gamma]$ with $V[G] = V[G_\gamma \times K \times H']$.

Replace H' with a $\mathbb{Q}$-generic H over $V[G_\gamma \times K]$ such that $V[G] = V[G_\gamma \times K \times H]$.

In $V[G]$, let $g : {}^{\kappa}\delta \to {}^{\kappa}\kappa$;

$$g(y) = \bigcup\{e_H(u) : u \subsetneq y\}.$$

Lemma 4

Let $y \in {}^{\kappa}\delta$.

1. $g(y)$ is $\mathrm{Add}(\kappa, 1)$-generic over $V[G_\gamma]$.

Sketch of the proof (the $\kappa > \omega$ case)

Let $K \times H'$ be $\mathbb{P}_\lambda \times \mathrm{Add}(\kappa, 1)$-generic over $V[G_\gamma]$ with $V[G] = V[G_\gamma \times K \times H']$.

Replace H' with a $\mathbb{Q}$-generic H over $V[G_\gamma \times K]$ such that $V[G] = V[G_\gamma \times K \times H]$.

In $V[G]$, let $g : {}^{\kappa}\delta \to {}^{\kappa}\kappa$;

$$g(y) = \bigcup\{e_H(u) : u \subsetneq y\}.$$

Lemma 4

Let $y \in {}^{\kappa}\delta$.

1. $g(y)$ is $\mathrm{Add}(\kappa, 1)$-generic over $V[G_\gamma]$.
2. $V[G]$ is a $\mathbb{P}_\lambda$-generic extension of $V[G_\gamma][g(y)]$.

Sketch of the proof (the $\kappa > \omega$ case)

Let $K \times H'$ be $\mathbb{P}_\lambda \times \mathrm{Add}(\kappa, 1)$-generic over $V[G_\gamma]$ with $V[G] = V[G_\gamma \times K \times H']$.

Replace H' with a $\mathbb{Q}$-generic H over $V[G_\gamma \times K]$ such that $V[G] = V[G_\gamma \times K \times H]$.

In $V[G]$, let $g : {}^\kappa\delta \to {}^\kappa\kappa$;

$$g(y) = \bigcup\{e_H(u) : u \subsetneq y\}.$$

Lemma 4

Let $y \in {}^\kappa\delta$.

1. $g(y)$ is $\mathrm{Add}(\kappa, 1)$-generic over $V[G_\gamma]$.
2. $V[G]$ is a $\mathbb{P}_\lambda$-generic extension of $V[G_\gamma][g(y)]$.
3. Therefore $\tau^{g(y)} \in X$.

Sketch of the proof (the $\kappa > \omega$ case)

Let $K \times H'$ be $\mathbb{P}_\lambda \times \mathrm{Add}(\kappa, 1)$-generic over $V[G_\gamma]$ with $V[G] = V[G_\gamma \times K \times H']$.

Replace H' with a $\mathbb{Q}$-generic H over $V[G_\gamma \times K]$ such that $V[G] = V[G_\gamma \times K \times H]$.

In $V[G]$, let $g : {}^\kappa\delta \to {}^\kappa\kappa$;

$$g(y) = \bigcup \{e_H(u) : u \subsetneq y\}.$$

Lemma 4

Let $y \in {}^\kappa\delta$.

1. $g(y)$ is $\mathrm{Add}(\kappa, 1)$-generic over $V[G_\gamma]$.
2. $V[G]$ is a $\mathbb{P}_\lambda$-generic extension of $V[G_\gamma][g(y)]$.
3. Therefore $\tau^{g(y)} \in X$.

Let $f : {}^\kappa\delta \to X$;

$$f(y) = \tau^{g(y)}.$$

f is a continuous map and is a homomorphism from $\mathbb{H}_\delta$ to R. (Item 3 in the definition of $\mathbb{Q}$ guarantees this).

The Hurewicz dichotomy for definable subsets of ${}^\kappa\kappa$

Let κ be an infinite cardinal such that $\kappa^{<\kappa} = \kappa$. Let $X \subseteq {}^\kappa\kappa$.

X is κ-compact iff every open cover of X has a subcover of size $<\kappa$.
X is K_κ iff X is the union of κ-many κ-compact sets.

X satisfies the Hurewicz dichotomy iff either X is contained in a K_κ subset of ${}^\kappa\kappa$ or there is a closed set $Y \subseteq X$ homeomorphic to ${}^\kappa\kappa$.

The Hurewicz dichotomy for definable subsets of ${}^\kappa\kappa$

Let κ be an infinite cardinal such that $\kappa^{<\kappa} = \kappa$. Let $X \subseteq {}^\kappa\kappa$.

X is κ-compact iff every open cover of X has a subcover of size $<\kappa$.
X is K_κ iff X is the union of κ-many κ-compact sets.

X satisfies the Hurewicz dichotomy iff either X is contained in a K_κ subset of ${}^\kappa\kappa$ or there is a closed set $Y \subseteq X$ homeomorphic to ${}^\kappa\kappa$.

Proposition

The Hurewicz dichotomy for X is implied by $\mathrm{OGD}^\kappa_\kappa(X, R)$ for the class of κ-dimensional box-open dihypergraphs R on X which are definable from an element of ${}^\kappa\mathrm{Ord}$.

The Hurewicz dichotomy for definable subsets of ${}^{\kappa}\kappa$

Let κ be an infinite cardinal such that $\kappa^{<\kappa} = \kappa$. Let $X \subseteq {}^{\kappa}\kappa$.

X is κ-compact iff every open cover of X has a subcover of size $<\kappa$.
X is K_κ iff X is the union of κ-many κ-compact sets.

X satisfies the Hurewicz dichotomy iff either X is contained in a K_κ subset of ${}^{\kappa}\kappa$ or there is a closed set $Y \subseteq X$ homeomorphic to ${}^{\kappa}\kappa$.

Proposition

The Hurewicz dichotomy for X is implied by $\mathrm{OGD}^{\kappa}_{\kappa}(X, R)$ for the class of κ-dimensional box-open dihypergraphs R on X which are definable from an element of ${}^{\kappa}\mathrm{Ord}$.

Corollary (Lücke, Motto Ros, Schlicht)

If $\lambda > \kappa$ is inaccessible, then in any $\mathrm{Col}(\kappa, {<}\lambda)$-generic extension $V[G]$, the Hurewicz dichotomy holds for all subsets $X \subseteq {}^{\kappa}\kappa$ which are definable from an element of ${}^{\kappa}\mathrm{Ord}$.

Questions

- Suppose $\kappa > \omega$. Is it consistent with ZFC that $\mathrm{OGD}^{\kappa}_{\kappa}(X)$ (i.e., for all box-open κ-dimensional dihypergraphs) holds for $\mathbf{\Sigma}^1_1(\kappa)$ subsets $X \subseteq {}^{\kappa}\kappa$?

Questions

- Suppose $\kappa > \omega$. Is it consistent with ZFC that $\mathrm{OGD}^{\kappa}_{\kappa}(X)$ (i.e., for all box-open κ-dimensional dihypergraphs) holds for $\mathbf{\Sigma}^1_1(\kappa)$ subsets $X \subseteq {}^{\kappa}\kappa$?
  For all subsets of ${}^{\kappa}\kappa$ which are definable using parameters in ${}^{\kappa}\mathrm{Ord}$?

Questions

- Suppose $\kappa > \omega$. Is it consistent with ZFC that $\mathrm{OGD}^{\kappa}_{\kappa}(X)$ (i.e., for all box-open κ-dimensional dihypergraphs) holds for $\mathbf{\Sigma}^1_1(\kappa)$ subsets $X \subseteq {}^{\kappa}\kappa$?
  For all subsets of ${}^{\kappa}\kappa$ which are definable using parameters in ${}^{\kappa}\mathrm{Ord}$?
- Which applications follow already from "$\mathrm{OGD}^{\omega}_{\omega}(X, R)$ for all definable R"?

Questions

- Suppose $\kappa > \omega$. Is it consistent with ZFC that $\mathrm{OGD}^{\kappa}_{\kappa}(X)$ (i.e., for all box-open κ-dimensional dihypergraphs) holds for $\mathbf{\Sigma}^1_1(\kappa)$ subsets $X \subseteq {}^{\kappa}\kappa$?
  For all subsets of ${}^{\kappa}\kappa$ which are definable using parameters in ${}^{\kappa}\mathrm{Ord}$?
- Which applications follow already from "$\mathrm{OGD}^{\omega}_{\omega}(X, R)$ for all definable R"?

  Conjecture: all of them do.

Questions

- Suppose $\kappa > \omega$. Is it consistent with ZFC that $\mathrm{OGD}^{\kappa}_{\kappa}(X)$ (i.e., for all box-open κ-dimensional dihypergraphs) holds for $\mathbf{\Sigma}^1_1(\kappa)$ subsets $X \subseteq {}^{\kappa}\kappa$?
  For all subsets of ${}^{\kappa}\kappa$ which are definable using parameters in ${}^{\kappa}\mathrm{Ord}$?
- Which applications follow already from "$\mathrm{OGD}^{\omega}_{\omega}(X, R)$ for all definable R"?

  Conjecture: all of them do.
- Which other applications of $\mathrm{OGD}^{\omega}_{\omega}(X)$ can be generalized to the setting of κ-Baire spaces for $\kappa > \omega$?

Questions

- Suppose $\kappa > \omega$. Is it consistent with ZFC that $\mathrm{OGD}_{\kappa}^{\kappa}(X)$ (i.e., for all box-open κ-dimensional dihypergraphs) holds for $\mathbf{\Sigma}_1^1(\kappa)$ subsets $X \subseteq {}^{\kappa}\kappa$?
  For all subsets of ${}^{\kappa}\kappa$ which are definable using parameters in ${}^{\kappa}\mathrm{Ord}$?
- Which applications follow already from "$\mathrm{OGD}_{\omega}^{\omega}(X, R)$ for all definable R"?

  Conjecture: all of them do.
- Which other applications of $\mathrm{OGD}_{\omega}^{\omega}(X)$ can be generalized to the setting of κ-Baire spaces for $\kappa > \omega$?
- OGA_{κ}: if $X \subseteq {}^{\kappa}\kappa$ and G is an open graph on X, then either G has a κ-coloring or G includes a complete subgraph of size κ^{+}.

Questions

- Suppose $\kappa > \omega$. Is it consistent with ZFC that $\mathrm{OGD}^{\kappa}_{\kappa}(X)$ (i.e., for all box-open κ-dimensional dihypergraphs) holds for $\mathbf{\Sigma}^1_1(\kappa)$ subsets $X \subseteq {}^{\kappa}\kappa$?

  For all subsets of ${}^{\kappa}\kappa$ which are definable using parameters in ${}^{\kappa}\mathrm{Ord}$?

- Which applications follow already from "$\mathrm{OGD}^{\omega}_{\omega}(X, R)$ for all definable R"?

  Conjecture: all of them do.

- Which other applications of $\mathrm{OGD}^{\omega}_{\omega}(X)$ can be generalized to the setting of κ-Baire spaces for $\kappa > \omega$?

- OGA_{κ}: if $X \subseteq {}^{\kappa}\kappa$ and G is an open graph on X, then either G has a κ-coloring or G includes a complete subgraph of size κ^+.

  Is OGA_{κ} consistent when $\kappa > \omega$?

Questions

- Suppose $\kappa > \omega$. Is it consistent with ZFC that $\mathrm{OGD}^{\kappa}_{\kappa}(X)$ (i.e., for all box-open κ-dimensional dihypergraphs) holds for $\mathbf{\Sigma}^1_1(\kappa)$ subsets $X \subseteq {}^{\kappa}\kappa$?
  For all subsets of ${}^{\kappa}\kappa$ which are definable using parameters in ${}^{\kappa}\mathrm{Ord}$?
- Which applications follow already from "$\mathrm{OGD}^{\omega}_{\omega}(X, R)$ for all definable R"?

  Conjecture: all of them do.
- Which other applications of $\mathrm{OGD}^{\omega}_{\omega}(X)$ can be generalized to the setting of κ-Baire spaces for $\kappa > \omega$?
- OGA_{κ}: if $X \subseteq {}^{\kappa}\kappa$ and G is an open graph on X, then either G has a κ-coloring or G includes a complete subgraph of size κ^{+}.

  Is OGA_{κ} consistent when $\kappa > \omega$? If so, how does it influence the structure of the κ-Baire space?

Thank you!