Miller Forcing with Canjar Ultrafilters

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Finite Strictly Increasing Sequences – Blocks

Definition

The set of finite strictly increasing sequences of natural numbers is called $\omega^{\uparrow<\omega}$.

The length of $s \in \omega^{\uparrow<\omega}$, $|s|$, is its domain.

For $s, t \in \omega^{<\omega}$, we say "$t$ extends s" or "s is an initial segment of t" and write $s \trianglelefteq t$ if $\mathrm{dom}(s) \subseteq \mathrm{dom}(t)$ and $s = t \restriction \mathrm{dom}(s)$.

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For $s, t \in \omega^{\uparrow<\omega}$ we have

$$s \trianglelefteq t \Leftrightarrow \mathrm{range}(s) \sqsubseteq \mathrm{range}(t)$$

and vice versa, going from finite subsets of ω to their increasing enumerations.

Definition

A subset $p \subseteq \omega^{\uparrow<\omega}$ that is closed under initial segments is called a tree.

The elements of a tree are called nodes.

A node $s \in p$ is called a splitting node of p if s has more than one direct $\vartriangleleft$-successor in p and *ω-splitting node of p* if s has infinitely many direct $\vartriangleleft$-successors in p. The set of splitting nodes of p is denoted by $\operatorname{sp}(p)$ while $\omega\text{-}\operatorname{sp}(p)$ denotes the set of ω-splitting nodes of p.

Special Sets of Blocks

Definition

(1) For any set A we write $[A]^{<\omega} = \{t : t \subseteq A, |t| < \omega\}$. The elements of $\mathrm{Fin} = [\omega]^{<\omega} \setminus \{\emptyset\}$ are called *blocks*.

(2) Let $\mathcal{F}$ be a filter over ω. We let

$$\mathcal{F}^{<\omega} = \{[A]^{<\omega} \setminus \{\emptyset\} : A \in \mathcal{F}\}$$

$$(\mathcal{F}^{<\omega})^{+} = \{B \subseteq \mathrm{Fin} : \forall A \in \mathcal{F}([A]^{<\omega} \cap B \neq \emptyset)\}$$

Guzmán and Kalajdzievski introduced a family of Miller forcings $\mathbb{PT}(\mathcal{F})$, $\mathcal{F}$ a filter over ω, extending the Fréchet filter.

Definition

Let $\mathcal{F}$ be a filter over ω. The forcing $\mathbb{PT}(\mathcal{F})$ consists of all $p \subseteq \omega^{\uparrow<\omega}$ such that for each $s \in p$ there is $t \trianglerighteq s$, such that $t \in \omega\text{-}\operatorname{sp}(p)$ and

$$\operatorname{sucspl}_p(t) := \{\operatorname{range}(r) \setminus \operatorname{range}(t) \ : \ r \text{ a } \triangleleft\text{-minimal infinitely splitting node of } p \text{ above } t\} \in (\mathcal{F}^{<\omega})^+.$$

Families of Superperfect Trees

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Such a t is called an $\mathcal{F}$-splitting node. We furthermore require of p that each ω-splitting node is an $\mathcal{F}$-splitting node and there is a unique $\triangleleft$-minimal ω-splitting node called the trunk of p, $\operatorname{tr}(p)$. The set of $\mathcal{F}$-splitting nodes of p is denoted by $\mathcal{F}\text{-}\operatorname{sp}(p)$.

Plain slide for a sketch

Lemma

The forcing $\mathbb{PT}(\mathcal{F})$ has the pure decision property.

$p \in \mathbb{PT}(\mathcal{F})$, $s \in \mathcal{F}\text{-}\operatorname{sp}(p)$, $D \subseteq \mathbb{PT}(\mathcal{F})$ open dense. Then

$$E(p, s, D) = \sqsubseteq -\min\{\operatorname{range}(t) \setminus \operatorname{range}(s) : \exists q \leq p(\operatorname{tr}(q) = t \wedge q \in D)\}$$

is in $(\mathcal{F}^{<\omega})^+$.

Let $f \in \mathcal{F}$. $p \upharpoonright\upharpoonright F$ contains only those nodes $t \in p$ for which $\operatorname{range}(t) \setminus \operatorname{tr}(p) \subseteq F$.

We have $p \upharpoonright\upharpoonright F \leq_0 p$.

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is in $(\mathcal{F}^{<\omega})^+$.
Let $f \in \mathcal{F}$. $p \restriction\restriction F$ contains only those nodes $t \in p$ for which $\operatorname{range}(t) \setminus \operatorname{tr}(p) \subseteq F$.
We have $p \restriction\restriction F \leq_0 p$.

Forcing with F_σ-Filters

Definition

(1) The partial order $\mathbb{F}_\sigma$ is the forcing with F_σ-filters over ω. Stronger filters are superfilters.

(2) If $\mathcal{F}$ is a filter, then $\mathbb{F}_\sigma(\mathcal{F})$ is the forcing with F_σ-filters that are compatible with $\mathcal{F}$, i.e. $\mathscr{G} \in \mathbb{F}_\sigma(\mathcal{F})$ iff $\mathscr{G}$ is an F_σ-filter and $\mathscr{G} \subseteq \mathcal{F}^+ = \{X \subseteq \omega : \forall(F \in \mathcal{F})(X \cap F \neq \emptyset)\}$.

Definition

Let G be an $\mathbb{F}_\sigma(\mathcal{F})$-generic filter. We let $\mathcal{U}$ be a $\mathbb{F}_\sigma(\mathcal{F})$-name for the union of G. By a density argument, the poset $\mathbb{F}_\sigma(\mathcal{F})$ forces that $\mathcal{U}$ is an ultrafilter that contains $\mathcal{F}$ as a subset.

Canjar Filters

Definition

$\mathcal{F}$ is called a Canjar filter if for any sequence $\langle X_n : n < \omega \rangle$ of elements of $(\mathcal{F}^{<\omega})^+$ there is a sequence $Y_n \in [X_n]^{<\omega}$ such that $\bigcup\{Y_n : n < \omega\} \in (\mathcal{F}^{<\omega})^+$.

Hrušák and Minami showed: A filter is Canjar iff Mathias forcing with second components in the filter does not add a dominating real.

More equivalent formulations are given by Blass Hrusak Verner, Chodounsky Repovs Zdomskyy, Guzmán Hrušák Martinez.

Generic Canjar Ultrafilters

Lemma

Canjar. The generic filter $\mathcal{U}$ of the forcing $\mathbb{F}_\sigma$ is such that Mathias forcing with it does not add a dominating real.

Another important concept is the following.

Definition

(1) A function $h\colon \omega \to \omega$ is called finite-to-one if for any n, the preimage of $\{n\}$, i.e. $h^{-1}[\{n\}]$, is finite (this includes the possibility of being empty).

(2) Let $\mathcal{F}$ and $\mathcal{U}$ be ultrafilters over ω. $\mathcal{F}$ and $\mathcal{U}$ are called nearly coherent if there is a finite-to-one function h such that $h(\mathcal{F}) = h(\mathcal{U})$ where $h(\mathcal{U}) = \{X \subseteq \omega : h^{-1}[X] \in \mathcal{U}\}$.

(3) A filter $\mathcal{F}$ is called almost ultra if there is a finite-to-one mapping h such that $h(\mathcal{F})$ is an ultrafilter.

$\mathbb{PT}(\mathcal{U})$ and Other Ultrafilters, II

Lemma

Let $\mathcal{W}$ be a P-point.

(a) If $\mathcal{U}$ is a Canjar ultrafilter that is not nearly coherent to $\mathcal{W}$, then forcing with $\mathbb{PT}(\mathcal{U})$ preserves $\mathcal{W}$.

(b) If a Canjar filter $\mathcal{F}$ is not almost ultra (see Def. 2.3(3)), then $\mathbb{F}_\sigma(\mathcal{F}) * \mathbb{PT}(\underset{\sim}{\mathcal{U}})$ preserves $\mathcal{W}$.

Definition

Let $f : \omega \to \omega$ be a strictly increasing function with $f(0) = 0$. A condition $p \in \mathbb{PT}(\mathcal{F})$ is said to have f-*blockstructure* if

$$(\forall t \in \mathcal{F}\text{-}\operatorname{sp}(p))(\forall r \in \operatorname{sucspl}_p(t))$$
$$(\exists k \in \omega)(\operatorname{range}(r) \setminus \operatorname{range}(t) \subseteq [f(k), f(k+1))). \tag{0.1}$$

Back to Mathias Forcing or the Canjar Game

Lemma

Let $\mathcal{F}$ be Canjar and $p \in \mathbb{PT}(\mathcal{F})$. There is an $f \in \omega^{\uparrow\omega}$ with $f(0) = 0$ and there is a $q \leq_0 p$ with f-blockstructure.

Predecessor lemma by Guzmán and Kalajdzievski

Lemma

Let $\mathcal{F}$ be Canjar and $\langle X_n : n < \omega \rangle$, $X_n \in (\mathcal{F}^{<\omega})^+$. There is an $f \in \omega^{\uparrow\omega}$ with $f(0) = 0$ such that $\bigcup \{X_n \cap \mathcal{P}(f(n)) : n < \omega\} \in (\mathcal{F}^{<\omega})^+$.

Premise

Theorem

Let $\alpha \leq \omega_1$ and let $\mathbb{P} = \langle \mathbb{P}_\gamma, \mathbb{Q}_\beta : \gamma \leq \alpha, \beta < \alpha \rangle$ be defined by induction on $\alpha \leq \omega_1$ as follows:

(1) $\mathbb{P}_0 = \{0\}$, *and*

(2) For $\beta < \alpha$ we have: If

- *for $\gamma < \beta$, r_γ is the $\mathbb{PT}(\mathcal{U}_\gamma)$-generic real over $\mathbf{V}^{\mathbb{P}_\gamma * \mathbb{F}_\sigma(\mathcal{F}_\gamma)}$,*
- $\mathbb{P}_\beta \Vdash \mathcal{F}_\beta = \mathrm{filter}(\{\mathrm{range}(r_\gamma) : \gamma < \beta\})$, *and*
- *$\mathcal{U}_\beta$ the $\mathbb{F}_\sigma(\mathcal{F}_\beta)$-generic filter over $\mathbf{V}^{\mathbb{P}_\beta}$,*

then $\mathbb{P}_\beta \Vdash \mathbb{Q}_\beta = \mathbb{F}_\sigma(\mathcal{F}_\beta) * \mathbb{PT}(\mathcal{U}_\beta)$.

(3) $\mathbb{P}_\alpha \Vdash \mathcal{F}_\alpha = \mathrm{filter}(\{\mathrm{range}(r_\gamma) : \gamma < \alpha\})$.

Then

the following holds

(A) $\mathbb{P}_\alpha$ is proper and forcing with $\mathbb{P}_\alpha$ preserves any P-point in $\bigcup\{\mathbf{V}^{\mathbb{P}_\beta} : \beta < \alpha\}$. For $\alpha < \omega_2$, we have $|\mathbb{P}_\alpha| \leq \aleph_1$.

(B) For any $\beta < \alpha$ if $\mathrm{cf}(\beta) \leq \omega$ then

$$\mathbb{P}_\beta \Vdash \mathcal{F}_\beta \text{ is not nearly ultra.}$$

and

$$\mathbb{P}_\beta * \mathbb{F}_\sigma(\mathcal{F}_\beta) \Vdash \mathcal{U}_\beta \text{ is a Canjar ultrafilter}$$

$$\text{and not nearly coherent } \mathcal{W} \text{ for any } \mathcal{W} \in \mathbf{V}^{\mathbb{P}_\beta}.$$

More Conclusions

(C) Let $\alpha = \omega_1$.

$\mathbb{P}_\alpha \Vdash \mathcal{F}_\alpha = \mathcal{U}_\alpha$ is a Canjar ultrafilter

and not nearly coherent any P-point $\mathcal{W} \in \bigcup\{\mathbf{V}^{\mathbb{P}_\gamma} : \gamma < \alpha\}$.

(D) $\forall \gamma < \beta < \alpha$, $\mathbb{P}_{\beta+1} \Vdash r_\beta \subseteq^* r_\gamma$.

Some Steps of the Proof

We prove the lemma by induction on α.

First suppose that $\alpha \leq \omega_2$ is a limit ordinal and the lemma is proved for $\gamma < \alpha$. For conclusion (A) we cite:

Theorem

(Blass, Shelah) If $\mathcal{W}$ is a P-point, α is a limit ordinal and $\mathbb{P}_\alpha = \langle \mathbb{P}_\gamma : \gamma < \alpha \rangle$ is the countable support limit and for $\gamma < \alpha$, the forcing $\mathbb{P}_\gamma$ is proper and preserves $\mathcal{W}$, then $\mathbb{P}_\alpha$ is proper and preserves $\mathcal{W}$.

Also the statement on the size of the forcing order is in the proper forcing book.

About the Ranges of the Tree Nodes

Lemma

Let $\mathcal{F}$ be Canjar and $p \in \mathbb{PT}(\mathcal{F})$. There is an $f \in \omega^{\uparrow\omega}$ with $f(0) = 0$ and there is a $q \leq_0 p$ with f-blockstructure.

$\mathcal{F}$ is Canjar iff I does not have a Winning Strategy in the Canjar Game for $\mathcal{F}$

Definition

For a filter $\mathcal{F}$ we consider the *Canjar Game* $\mathscr{G}(\mathcal{F})$. Player I and player II alternately play sets $X_0, Y_0, X_1, Y_1, \ldots$. The rules are $X_i \in (\mathcal{F}^{<\omega})^+$, $Y_i \in [X_i]^{<\omega} \setminus \{\emptyset\}$ for every $i \in \omega$.

I	X_0		X_1		X_2		$\ldots$
II		Y_0		Y_1		Y_2	

After ω rounds, player II wins if $\bigcup_{n\in\omega} Y_n \in (\mathcal{F}^{<\omega})^+$.

A Lemma about Directions

Lemma

Let $\mathcal{W}$ be a P-point.
If $\mathcal{U}$ is a Canjar ultrafilter that is not nearly coherent to $\mathcal{W}$, then forcing with $\mathbb{PT}(\mathcal{U})$ preserves $\mathcal{W}$.

Fresh Directions after Half-Steps

Lemma

If a filter $\mathcal{F}$ is not almost ultra (see Def. 2.3(3)) and $\mathbb{F}_\sigma(\mathcal{F}) \Vdash \underset{\sim}{\mathcal{U}}$ is Canjar, then $\mathbb{F}_\sigma(\mathcal{F}) * \mathbb{PT}(\underset{\sim}{\mathcal{U}})$ preserves $\mathcal{W}$.

Definition

Let $X \subseteq \mathrm{Fin}$. We let

$$C(X) = \{A \subseteq \omega : \forall s \in X (s \cap A \neq \emptyset)\}.$$

Lemma by Guzman and Kalajdziesvky

Let $\mathscr{G}$ be a filter. $\mathcal{F} \Vdash_{\mathbb{F}_\sigma(\mathscr{G})} X \in (\mathcal{U}(\underset{\sim}{\mathscr{G}})^{<\omega})^+$ iff $C(X) \subseteq \mathrm{filter}(\mathcal{F} \cup \mathscr{G})$

"$\Rightarrow$" Let $H \notin \mathrm{filter}(\mathcal{F} \cup \mathscr{G})$. Then H^c is $\mathrm{filter}(\mathcal{F} \cup \mathscr{G})$-positive and $\mathcal{F} \geq \mathcal{F} \cup \{H^c\}$ is a condition in $\mathbb{F}_\sigma(\mathscr{G})$. Then $\exists s \in X, s \subseteq H^c$. Thus $H \notin C(X)$.

"$\Leftarrow$" Suppose $C(X) \subseteq \mathrm{filter}(\mathcal{F} \cup \mathscr{G})$. Then $\forall A \in C(X)$, $A^c \notin \mathcal{U}(\mathscr{G})$. Hence for any $D \in \mathcal{U}(\mathscr{G})$, $D^c \notin C(X)$ and hence $\exists s \in X (s \subseteq D)$.

Lemma

Let $\alpha = \omega_1$. $\mathbb{P}_\alpha$ forces that $\mathcal{F}_\alpha$ is a Canjar ultrafilter that is not nearly coherent to any P-point in $\bigcup_{\gamma<\alpha} V^{\mathbb{P}_\gamma}$.

An easy density argument shows that for $\mathrm{cf}(\alpha) = \omega_1$, $\mathbb{P}_\alpha \Vdash \mathcal{F}_\alpha = \mathcal{U}_\alpha$ is ultra since any name for a subset of ω appears in some $V^{\mathbb{P}_\gamma}$, $\gamma < \alpha$.

Any name h for a finite-to-one function appears at some $\mathbb{P}_\beta$, $\beta < \alpha$. Let $\mathcal{W}$ be also in $\mathbf{V}^{\mathbb{P}_\beta}$. Then $\mathbb{P}_{\beta+1} \Vdash \exists X \in \mathcal{U}_\beta \exists Y \in \mathcal{W} h[X] \cap h[Y] = \emptyset$ and by Lemma VII, 7.13b Kunen this is preserved upwards. Hence $\mathbb{P}_\alpha$ forces that $\mathcal{U}_\alpha$ is not nearly coherent to any P-point in $V^{\mathbb{P}_\gamma}$, $\gamma < \alpha$.

Sketch of a Proof

The only not so easy statement is: $\mathbb{P}_\alpha$ forces that $\mathcal{U}_\alpha$ is Canjar. By induction hypothesis we know that for $\beta < \alpha$ the name $\mathcal{U}_\beta$ is forced by $\mathbb{P}_\beta * \mathbb{F}_\sigma(\mathcal{F}_\beta)$ to be a Canjar ultrafilter. Every name for an ω-sequence of sets of blocks appears for the first time at an iteration stage of countable cofinality.

Suppose we have $p \in \mathbb{P}_\alpha$ and a $\mathbb{P}_\alpha$-name $\langle X_n : n < \omega \rangle$ such that

$$p \Vdash (\forall n)(X_n \in ((\mathcal{U}_\alpha)^{<\omega})^+).$$

Thus for some $\beta_0 < \alpha$, we have that $\langle X_n : n < \omega \rangle$ is equivalent to an $\mathbb{P}_{\beta_0}$-name. W.l.o.g., let $\langle X_n : n < \omega \rangle$ be a $\mathbb{P}_{\beta_0}$-name.

$$(p \restriction \beta_0, p_1(\beta_0)) \Vdash_{\mathbb{P}_{\beta_0 * \mathbb{F}_\sigma(\mathcal{F}_{\beta_0})}} (\forall n)(X_n \in ((\mathcal{U}_{\beta_0})^{<\omega})^+).$$

Since $\mathbb{P}_{\beta_0} * \mathbb{F}_\sigma(\mathcal{F}_{\beta_0})$ forces that $\mathcal{U}_{\beta_0}$ is Canjar, there is a $\mathbb{P}_{\beta_0}$-name for a sequence $\langle Y_n : n < \omega \rangle$ such that

$$(p \restriction \beta_0, p_1(\beta_0)) \Vdash_{\mathbb{P}_{\beta_0} * \mathbb{F}_\sigma(\mathcal{F}_{\beta_0})} (\forall n)(Y_n \in [X_n]^{<\omega} \wedge \bigcup\{Y_n : n < \omega\} \in ((\mathcal{U}_{\beta_0})^{<\omega})^+.$$

But then by the characterisation of $\mathbb{F}_\sigma$-forcing

$$(p \restriction \beta_0, p_1(\beta_0)) \Vdash_{\mathbb{P}_\alpha} (\forall n)(Y_n \in [X_n]^{<\omega} \wedge \bigcup\{Y_n : n < \omega\} \in ((\mathcal{U}_\alpha)^{<\omega})^+.$$

This is seen as follows: We assume that $q \geq p$, $q \in \mathbb{P}_\alpha$ and $q \Vdash Y \in \mathcal{U}_\alpha$. Again there is $\beta_1 < \alpha$ such that $q \in \mathbb{P}_{\beta_1}$ and Y is a $\mathbb{P}_{\beta_1}$-name and

$$(q \restriction \beta_1, q_1(\beta_1)) \Vdash_{\mathbb{P}_{\beta_1} * \mathbb{F}_\sigma(\mathcal{F}_{\beta_1}))} Y \in \mathcal{U}_{\beta_1}.$$

We assume that $\beta_1 \geq \beta_0$.

We let $Z = \bigcup\{Y_n : n < \omega\}$. By the characterisation of $\mathbb{F}_\sigma$-forcing we have in $\mathbf{V}^{\mathbb{P}_{\beta_0}\restriction(p\restriction\beta_0)}$,

$$C(Z) \subseteq \text{filter}(p_1(\beta_0) \cup \{r_\gamma : \gamma \in \beta_0\}).$$

This is a Π^1_1-relation of Z and p, and hence holds, again by Lemma VII, 7.13b Kunen, also in $\mathbf{V}^{\mathbb{P}_{\beta_1}\restriction(q\restriction\beta_1)}$. So in $\mathbf{V}^{\mathbb{P}_{\beta_1}\restriction(p\restriction\beta_0)}$,

$$C(Z) \subseteq \text{filter}(p_1(\beta_0) \cup \{r_\gamma : \gamma \in \beta_0\}).$$

In the same model we can increase the filter as follows:

$$C(Z) \subseteq \text{filter}(q_1(\beta_1) \cup \{r_\gamma : \gamma \in \beta_1\}).$$

End of Proof

Hence by the characterisation of $\mathbb{F}_\sigma$ forcing, $q \restriction \beta_1 \Vdash_{\mathbb{P}_{\beta_1} * \mathbb{F}_\sigma(\mathcal{F}_{\beta_1})} Z \in (\mathcal{U}_{\beta_1}^{<\omega})^+$, and since q and Y and β_1 were arbitrary, and we are done.

Thank you!