Mathematics in Metrical form: Its pros and cons

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Culture of Mathematics - !V, INSA

Outline

- Introduction
  - The need for metrical composition
  - A few examples
  - The pros of having such compositions
- The assessment made by historians (absence of proofs)
  - What constitutes a proof?
  - Should we wash out differences and look deeper?
  - Striking unanimity amongst the historians
  - The cons of such assesment
- How far is the assessment of historians correct?
- The proof of the surface area
- The Infinite series π given by Mādhava (14th century)
- Proof of the series
- Concluding remarks

Introduction

- A interesting feature of Indian Mathematics is that it has been composed in the form of poetry.
  - Is it by choice / compulsion ? (perhaps both!)
  - The Indian mathematicians have successfully managed to couch a variety of fomulae, even those that were quite involved, in the form of beautiful verses.
- Starting with
  - representation of numbers ($khyugh\d{r}$ = 4.32 mil; Rama = 52)
  - to presenting the various mensuration formulae
  - to stating certain useful algebraic identities
  - to delineating the procedure for solving quadratic, and indeterminate equations (I and II order)
  - to specifying the value of π
  - to representing it in the form of an infinite series,
  - to giving the derivative of ratio of two functions,

  all of them have been couched in metrical form (verses).

The surface area and the volume of a sphere

वृत्तक्षेत्रे परिधिगुणितव्यासपादः फलं तत्
क्षुण्णं वेदैरुपरि परितः कन्दुकस्येव जालम्।
गोलस्यैवं तदपि च फलं पृष्ठजं व्यासनिघ्नं
षड्भिर्भक्तं भवति नियतं गोलगर्भे घनाख्यम्॥

$$\begin{aligned}
A_{circle} &= \text{circumference} \times \text{a quarter of diameter} \\
&= 2\pi r \times D/4 \\
&= \pi r^2 \\
A_{sphere} &= A_{circle} \times veda \\
&= 4\pi r^2 \\
V_{sphere} &= A_{sphere} \times D/6 \\
&= \frac{4}{3}\pi r^3
\end{aligned}$$

The demonstration of these results are found in his commentary on *Siddhāntaśiromaṇi* (*Bhuvanakośa*) called *Vāsanābhāṣya*.

Theorem on the square of chords

In his *Āryabhaṭīya*, Āryabhaṭa has presents the theorem on the product of chords as follows (in half *āryā*):

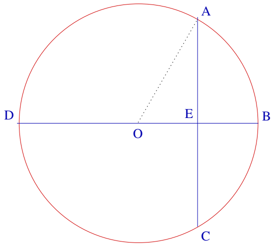
वृत्ते शरसंवर्गः अर्धज्यावर्गः स खलु धनुषोः ॥

(Āryabhaṭīya, Gaṇita 17)

- The words *varga* and q*samvarga* refer to square and product respectively.
- Similarly, *dhanus* and *śara* refer to arc and arrow respectively.

Using modern notations the above *nyāya* may be expressed as:

$$\text{product of } \textit{śaras} = \text{Rsine}^2$$
$$DE \times EB = AE^2$$



Problem involving solution of a quadratic equation

Example drawn from *Itihāsa*

पार्थः कर्णवधाय मार्गणगणं क्रुद्धो रणे सन्दधे
तस्यार्धेन निवार्य तच्छरगणं मूलैश्चतुर्भिर्हयान्।
शल्यं षड्भिः अथेषुभिस्त्रिभिरपि छत्रं ध्वजं कार्मुकं
चिच्छेदास्य शिरश्शरेण कति ते यानर्जुनः सन्दधे॥

- पार्थः, कर्णवधाय – Arjuna in order to slay Karṇa
- मार्गणगणम् – a quiver of arrows
- तस्यार्धेन निवार्य – Avoiding with half of them
- मूलैश्चतुर्भिर्हयान् – [He killed] all his horses with four times the square root of the arrows.
- छत्रं ध्वजं कार्मुकं – used one arrow each . . . , umbrella, flag, and bow.
- कति ते यानर्जुनः सन्दधे – How many arrows did Arjuna discharge totally?

Problem involving solution of a quadratic equation

Example drawn from *Itihāsa*

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चिच्छेदास्य शिरश्शरेण कति ते यानर्जुनः सन्दधे॥

- Let 'x' denote the total number of arrows discharged by Arjuna.
- As per the information provided in the verse,

$$\frac{1}{2} + 4\sqrt{x} + 6 + 3 + 1 = x \tag{1}$$

- This reduces to the quadratic equation:

$$x - 8\sqrt{x} - 20 = 0, \tag{2}$$

whose solutions are $\sqrt{x} = (10, \ -2)$. Of the two, we can only consider the $+$ve value, since the no. of arrows shot cannot be $-$ve. Hence, $x = 100$ is the answer.

Honey-bee problem (solution to quadratic; ex. from nature)

अलिकुलदलमूलं मालतीं यातमष्टौ
निखिलनवमभागः शालिनीं भृङ्गमेकम् ।
निशि परिमललुब्धं पद्ममध्ये निरुद्धं
प्रतिरणति रणन्तं ब्रूहि कान्तेऽलिसङ्ख्याम् ॥

From a group of bees (x), square root of the half ($\sqrt{\frac{x}{2}}$) went to the *mālatī* tree. Again eight-ninth of the bees ($\frac{8}{9}x$) went to the *śālinī* tree. Of the remaining two, being absorbed in the fragrance [of the lotus], one got itself trapped inside the lotus he started moaning and wailing [from inside]. Responding to that, the beloved too moaned [from outside]. Now, tell me my beloved, the total number bees that were there.



Summation of series

सैकपदघ्नपदार्धमथैका-
द्यङ्कयुतिः किल सङ्कलिताख्या।
सा द्वियुतेन पदेन विनिघ्नी
स्यात् त्रिहृता खलु सङ्कलितैक्यम्॥

$$V_n^{(1)} = \frac{n\,(n+1)}{2}\,. \tag{3}$$

The second summation (*dvitīya-saṅkalita*) is given by

$$V_n^{(2)} = \frac{n(n+1)(n+2)}{1.2.3} \tag{4}$$

$$\begin{aligned} V_n^{(0)} &= 1 + 1 + \ldots + 1 = n \\ V_n^{(1)} &= V_1^{(0)} + \ldots + V_n^{(0)} = 1 + 2 + \ldots + = \frac{n(n+1)}{2} \\ V_n^{(2)} &= V_1^{(1)} + V_2^{(1)} + \ldots + V_n^{(1)} = \frac{n(n+1)(n+2)}{1.2.3} \end{aligned}$$

A practical application of the formula (population growth?)

प्रतिवर्षं गौः सूते वर्षत्रितयेन तर्णकीं तस्याः।
विद्वन् विंशतिवर्षैः गौरेकस्याश्च सन्ततिं कथय॥ (From GK)

Interesting examples in geometry

The bamboo problem: Comparing the two presentations

यदि समभुवि वेणुः द्वित्रिपाणिप्रमाणः
गणकपवावेगात् एकदेशे स भग्नः।
भुवि नृपमितहस्तेष्वेवलग्नं तदग्रं
कथय कतिषु मूलात्, एष भग्नः करेषु॥

The choice of the meter above is such that even as you read the phrase गणकपवावेगात्, you feel the sense of the speed of the wind. On the other hand, Bhāskara - I presents the problem as follows:

षोडशहस्तो वंशः पवनेन निपातितः स्वमूलात्।
अष्टौ गत्वा पतितः कस्मिन् भग्नो मरुत्वतो वाच्यः॥

A bamboo of sixteen *hastas* was made to fall by the wind. It fell such that its tip hit the ground at eight [*hastas*] from its root. Where was it broken by the possesser (Lord) of the wind, is to be said.

Bhāskara's description of the notion of Infinity

Bhāskara introduces the notion of infinity by resorting to a beautiful analogy from philosophy, fairly 'well-known' in society:

अस्मिन् विकारः खहरे न राशौ अपि प्रविष्टेष्वपि निस्सृतेषु।
बहुष्वपि स्यात् लयसृष्टिकाले, अनन्तेऽच्युते भूतगणेषु यद्वत्॥

- अस्मिन् राशौ – In this number
- खहरे – obtained by dividing by zero
- विकारः न – there is [absolutely] no change
- अपि प्रविष्टेषु – even when many entities enter into
- अपि निस्सृतेषु – even as many entities move out
- यद्वत् अनन्तेऽच्युते – As in the case of the Lord *Acyuta* (*Ananta*).

Invocatory verse – exemplifying the IMPORTANCE OF ALGEBRA

A beautiful example of दीपकालङ्कार (प्रस्तुताप्रस्तुतवर्णनम्)

उत्पादकं यत् प्रवदन्ति बुद्धेः अधिष्ठितं सत् पुरुषेण साङ्ख्याः।
व्यक्तस्य कृत्स्नस्य तदेकबीजं अव्यक्तम् ईशं गणितं च वन्दे॥

व्यक्तस्य कृत्स्नस्य तदेकबीजं – that entity alone is the cause (बीजं) for the entire manifest [universe] – This feature is applicable to all the three.

अधिष्ठितं सत् पुरुषेण – Here we need to describe carefully:

- S: The अव्यक्त = प्रकृति being directed by the पुरुष
- V: The ईश = सोपधिकं ब्रह्म whose substratum is the निरुपाधिब्रह्म
- M: The गणितं being expounded by an able mathematician

Zero and Infinity: शून्य and अनन्त

Charles Seife observes:[1]

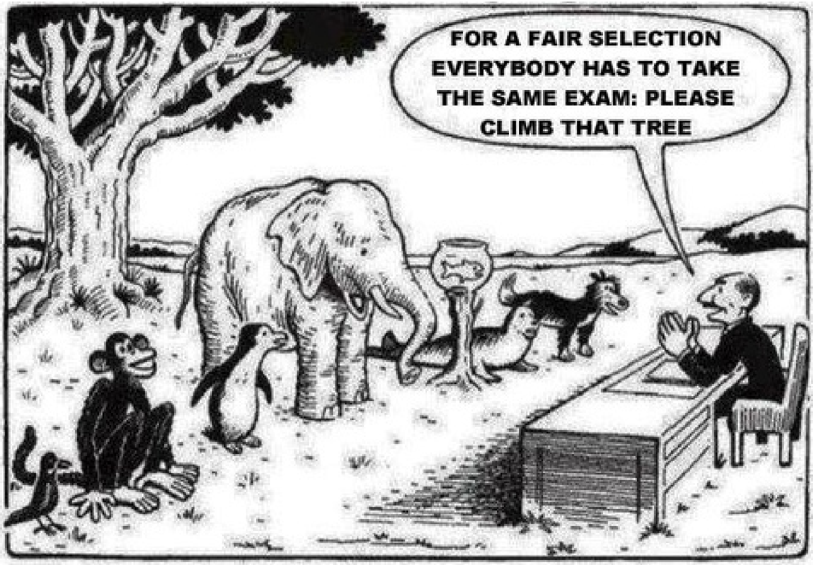
Unlike Greece, India never had the fear of the infinite or of the void. Indeed, it embraced them. . . . Indian mathematicians did more than simply accept zero. They transformed it changing its role from mere placeholder to number. The reincarnation was what gave zero its power. The roots of Indian mathematics are hidden by time. . . . Our numbers (the current system) evolved from the symbols that the Indians used; by rights they should be called Indian numerals rather than Arabic ones. . . . Unlike the Greeks the Indian did not see the squares in the square numbers or the areas of rectangles when they multiplied two different values. Instead, they saw the interplay of numerals—numbers stripped of their geometric significance. This was the birth of what we now know of *algebra*.

[1] *Zero:The Biography of a Dangerous Idea*, Rupa & Co. 2008, pp. 63–70.

The Pros

- When I learnt mathematics, the teacher would simply ‘teach’ a solution technique, and present a set of formula.
- We were expected to learn the technique, memorize the formulae, then work out those problems given at the end of the chapter, repeatedly; sometimes laboriously practice the application of the fomula/technique until mastery is achieved.
- I do not recall a single problem that could be related to practical life – as given in these ancient texts.
- The texts on Indian mathematics, soon after enunciating a rule or principle present plenty of examples drawn from day to day life – all in the form of beautiful verses.
- The act of memorizing besides generating fun could also be of help in developing certain ‘desirable’ mental faculty.

The current education system & perception imposed



The Cons

Morris Kline[2] observes:

- Sometimes the Hindus were aware that a formula was only approximately correct and sometimes they were not. Their values of π were generally inaccurate; ...
- They offered no geometric proofs; on the whole they cared little for geometry.
- In trigonometry the Hindus made a few minor advances[3] ...
- As our survey indicates, the Hindus were interested in and contributed to the arithmetical and computational activities of mathematics rather than to the deductive patterns. There is much good procedure and technical facility, but no evidence that they considered proof at all.[4]

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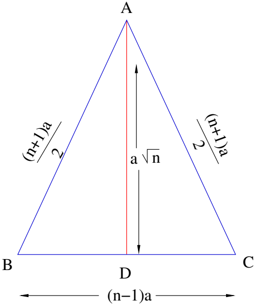
[2] Kline is credited with more than a dozen books on various aspects of mathematics such as history, philosophy, and pedagogy

[3] This observation is despite the fact that Hindus have extensively employed *Jīveparasparanyāya*, and obtained complicated results...

[4] Morris Kline, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, New York 1972, pp. 188-190.

To construct a square that is n times a given square

- Kātyāyana gives an ingenious method to construct a square whose area is n times the area of a given square.



यावत्प्रमाणानि समचतुरश्राणि एकीकर्तुं
चिकीर्षेत् एकोनानि तानि भवन्ति तिर्यक्।
द्विगुणान्येकत एकाधिकानि त्र्यस्रिर्भवति।
तस्येषुः तत्करोति। [*Kt.Sl* VI 7]

As much . . . one less than that forms the base . . . the arrow of that [triangle] makes that (gives the required number $\sqrt{n}$).

- In the Figure $BD = \frac{1}{2}BC = (\frac{n-1}{2})a$. Consider $\triangle ABD$,

$$\begin{aligned} AD^2 &= AB^2 - BD^2 = \left(\frac{n+1}{2}\right)a^2 - \left(\frac{n-1}{2}\right)a^2 \\ &= \frac{a^2}{4}\ (n+1)^2 - (n-1)^2 = \frac{a^2}{4} \times 4n = (na^2) \end{aligned}$$

What constitutes a proof?

- If one takes the definiton of proof to be simply

  > *an evidence or argument adduced in order to convince oneself/others regarding the truth of an assertion,*

  then it would certainly not be possible to make a categorical statement that Hindus NEVER considered proof at all.

- However, given the fact that history is replete with such statements, there should be some fundamental difference in the view point regarding *what constitutes of a proof?*.

- The difference perhaps stems from the fact that there is an 'appeal' to the empirical in convincing about the truth of an assertion in the Indian tradition, whereas there seems to be a 'rejection' of the empirical in the Platonic and the Neo-platonic tradition.

- This also to a certain extent has a bearing on the purpose defined in pursuing a certain discipline.

What constitutes a proof?

- According to the *Platonist* view, "proof is a mind-independent abstract object, eternal, unchanging, not located in space-time, and evidently causally inert."[5]
- It is evident from the above description that proof is "not" something that is created by a mathematician but is some abstract "ideal" thing that is discovered by him and hence *infallible* and *necessary truth*.
- In contrast to this, it may be mentioned here that *upapatti* – not necessarily employed in astronomy and mathematics alone, but used in the other branches of philosophy as well – is not considered as an mind-independent, infallible abstract object.
- It is as much a human construct (*puruṣa-buddhi-prabhava*) as any other thing. Hence *upapatti*-s are liable to be refined, improved and refuted too, unlike the eternal unchanging nature of proof conceived by Plato.

[5] Richard Tieszen, "What is Proof" in Michael Detlefson (ed), *Proof, Logic and Formalization*, Routledge, New York 1992, p. 58.

What constitutes a proof?

- Yuri I. Manin,[6] observes:

  > *The evolution of commonly accepted criteria for an argument's being a proof is an almost untouched theme in the history of science. . . .*[7]

- This criteria has been succinctly stated by Davis and Hersh in their recently published scholarly work.

  > *Abstraction, formalization, axiomatization, deduction – here are the ingredients of proof. And the proofs in modern mathematics, though they may deal with different raw material or lie at deeper level, have essentially the same feel to the student or the researcher as the one just quoted.*[8]

- Thus as far as the western tradition is concerned, the heart of the mathematical proof lies in axiomatization and deduction, though the laws of deduction are likely to be awesome in their complexity (may run to 40 dull pages) in certain cases.

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[6] A Russian mathematician, well known for work in algebraic geometry, Diophantine geometry, algorithms and mathematical logic.

[7] Yu. I. Manin, *A Course in Mathematical Logic*, Springer-Verlag, New York

Acceptance of the empirical $\Rightarrow$ The Epicurean Ass

- To Hilbert,[9] introducing the empirical was quite tricky:
  - if the appeal to the empirical is permissible in the proof of one theorem (Elements I.4), they why not permit it in the proof of *all* theorems?
  - why not permit translation, rotation of triangles in proving the "Pythagorean" theorem (as in *Yuktibhāṣā*)?
- If that were to be done most of the theorems become obvious and trivial!! Noting this, an particularly considering *Elements* I.20[10], the Epicureans (counterpart of *Lokāyatas*) seem to have argued with the followers of Euclid:

  > *Any ass knows the theorem, since the ass went straight to the hay, and does not take a circuitous route*

- To this Proclus seems to have said: the ass only *knew* that the theorem is true but does not know *why* it was true!.

[9] The current day mathematics is highly influenced by Hilbert's analysis of Euclid's *Elements*.

[10] The length of the any two sides of the triangle is greater than the third.

Unanimity amongst the historians of mathematics

There seems to be a striking unanimity in presenting the view that Hindu mathematics is bereft of proofs. The renowned historian C. B. Boyer observes:

- With the Hindus the view was different. They saw no essential unlikeliness between rectilinear and curvilinear figures, for each could be measured in terms of numbers; . . .
- The strong Greek distinction between the discreteness of number and the continuity of geometrical magnitude was not recognized, for it was superfluous to men who were not bothered by the paradoxes of Zeno or his dialectic.
- Questions concerning incommensurability, the infinitesimal, infinity, the process of exhaustion, and the other inquiries leading toward the conceptions and methods of calculus were neglected.[11]

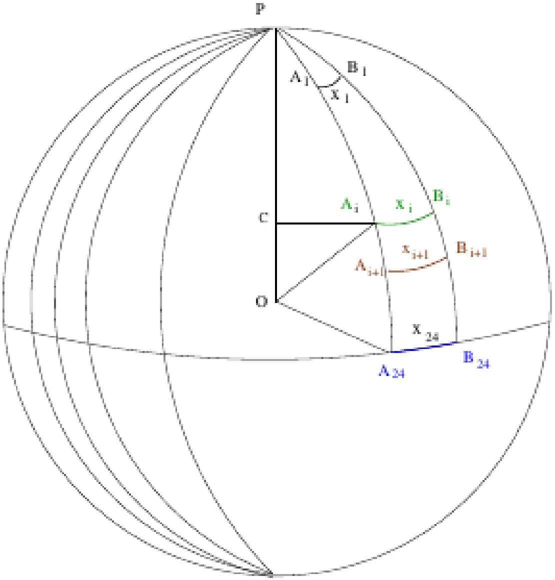
[11] C. B. Boyer, *The History of Calculus and its Conceptual Development,* Dover Publications, Inc., New York 1949, p. 62.

Demonstration of the surface area of a sphere

Bhāskara has demonstrated as to how one could derive the formula for the surface of a sphere by dividing the surface into lunes. The set of verses that present this demonstration in *Siddhāntaśiromani* are:

गोलस्य परिधिः कल्प्यो वेदघ्नज्यामितेर्मितः। (the no. of Rsines)
मुखबुध्नगरेखाभिः यद्वदामलके स्थिताः॥ (simile of gooseberry)
दृश्यन्ते वप्रकास्तद्वत् प्रागुक्तपरिधेर्मितान्। (similarly the lunes are seen)
ऊर्ध्वाधःकृतरेखाभिः गोले वप्रान् प्रकल्पयेत्॥
तत्रैकवप्रकक्षेत्रफलं खण्डैः प्रसाध्यते। (total area from elementary areas)
सर्वज्यैक्यं त्रिभज्यार्धहीनं त्रिज्यार्धभाजितम्॥
एवं वप्रफलं तत्स्याद्गोलव्याससमं यतः।
परिधिव्यासघातोऽतो गोलपृष्ठफलं स्मृतम्॥

Demonstration of the surface area of a sphere



Demonstration of the surface area of a sphere

- Let x_i be the length of the segment A_iB_i. Since the quadrant is divided into equal parts i.e., $A_iA_{i+1} = h$, the area of the i^{th} trapezium $A_iB_iB_{i+1}A_{i+1}$ is given by

$$A_{x_i} = \frac{1}{2}(x_i + x_{i+1}) \times h \tag{5}$$

- Now the area of the semi-lune is

$$A = \frac{1}{2}x_1 h + \sum_{i=1}^{23} \frac{1}{2}(x_i + x_{i+1})h \tag{6}$$

  where x_i and x_{i+1} form the face and base of the i^{th} trapezium.

- The next step is to expressing the above area in term of the Rsines.

Demonstration of the surface area of a sphere

By definition $A_{24}B_{24} = 1$. Hence by rule of three,

$$R : 1 \quad :: \quad r_i : x_i \tag{7}$$

$$\text{Therefore,} \qquad x_i \quad = \quad \frac{r_i}{R} \tag{8}$$

Using (8) in (6), we have

$$A = \frac{1}{R}\left[r_1 + r_2 + \ldots + r_{24} - \frac{1}{2}r_{24}\right] \times h \tag{9}$$

Since $h = 1$ and $r_{24} = R$, the above equation reduces to

$$A = \frac{1}{R}\left[r_1 + r_2 + \ldots + r_{24} - \frac{R}{2}\right] \tag{10}$$

Considering the triangle OCA_i, we have

$$\sin i\alpha \quad = \quad \frac{CA_i}{OA_i} = \frac{r_i}{R} \tag{11}$$

$$\text{or,} \qquad r_i \quad = \quad R \sin i\alpha \qquad (i^{th}\ jy\bar{a}) \tag{12}$$

Demonstration of the surface area of a sphere

Therefore, (10) becomes

$$A = \frac{1}{\text{trijyā}} \times \left[\text{Sum of Rsines} - \frac{\text{trijyā}}{2}\right] \tag{13}$$

The above is only area of a semi-lune. Since *vapraka* refers to the full lune, the area of the *vapraka* is given by

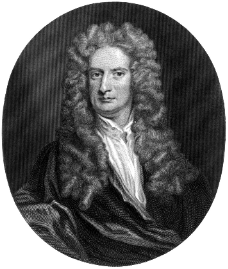
$$2A = \frac{\text{Sum of Rsines} - \frac{\text{trijyā}}{2}}{\frac{\text{trijyā}}{2}} \tag{14}$$

$$2A = \frac{54233 - 1719}{1719} = 30'33'' \tag{15}$$

This is the diameter of the sphere whose circumference is 96. Finally Bhāskara makes a crucial argument – परिधितुल्यकाश्च वप्रका इति परिधिव्यासघातो गोलपृष्ठफलमित्युपपन्नम्।

The current history of Calculus

- The genesis and evolution of calculus is indeed fascinating story that speaks of the genius and proficiency of various characters involved in it.
- Unfortunately we do not have a proper narration. The legends currently available are neither “truthful” nor “complete”.
- This is so not necessarily because of the lack of knowledge (ignorance), which can be easily condoned or pardoned!.
- Normally while speaking of calculus only two names come up – Newton and Leibniz;



Newton

Leibniz

Signal Contribution of Newton

Mathematical achievements that led to the launch of the calculus

- Some of the landmark contribution of Newton to mathematics are to be found in his "early" work *De analysi*. This includes:
    1. finding the generalized binomial expansion , which turn certain expression into infinite series
    2. finding technigues for inverting series
    3. inverting the sine series through the quadrature of simple curves
- It is very interesting to note that, when confronted with complicated expression, Newton tried the "reduce" (expand) it into an infinite series and then sum the result
- This led him to derive a "mathematical blockbuster" – The infinite series for sine of an angle. It has been exclaimed by William Dunham:

    > The early Newton tends to surpass the mature work of just about anyone else.

Signal Contribution of Leibniz

- The amount of praises that were showered upon Leibniz of discovering the series,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots$$

  was indeed phenominal. Why?

- Because "it was proved for the first time that the area of a circle was exactly equal to a series of rational quantities".
- Dividing both sides of the above series by two and grouping the terms, it can be easily seen that it reduces to

$$\frac{\pi}{8} = \frac{1}{2^2 - 1} + \frac{1}{6^2 - 1} + \frac{1}{10^2 - 1} + \frac{1}{14^2 - 1} + \dots$$

- Be that as it may! Was Leibniz the first to discover the series?

Different approximations to π

- The *Śulba-sūtra*-s, give the value of π close to 3.088.
- Āryabhaṭa (499 AD) gives an approximation which is correct to four decimal places.

चतुरधिकं शतमष्टगुणं द्वाषष्टिस्तथा सहस्राणाम्।
अयुतद्वयविष्कम्भस्य 'आसन्नो' वृत्तपरिणाहः॥

$$\pi \approx \frac{(100 + 4) \times 8 + 62000}{20000} = \frac{62832}{20000} = 3.1416$$

- Then we have the verse of *Līlāvatī*[12]

व्यासे भनन्दाग्निहते विभक्ते खबाणसूर्यैः परिधिः सुसूक्ष्मः ।
द्वाविंशतिघ्ने विहृतेऽथ शैलैः स्थूलोऽथवा स्याद् व्यवहारयोग्यः ॥

$$\pi = \frac{3927}{1250} = 3.1416 \qquad \text{that's same as Āryabhaṭa's value.}$$

[12] *Līlāvatī* of Bhāskarācārya, verse 199.

Different approximations to π

The commentary *Kriyākramakarī* further proceeds to present more accurate values of π given by different *Ācāryas*.

माधवाचार्यः पुनः अतोप्यासन्नतमां परिधिसङ्ख्यामुक्तवान् –

विबुधनेत्रगजाहिहुताशनत्रिगुणवेदभवारणबाहवः ।
नवनिखर्वमिते वृतिविस्तरे परिधिमानमिदं जगदुर्बुधाः ॥[13]

The values of π given by the above verses are:

$$\pi = \frac{2827433388233}{9 \times 10^{11}} = 3.141592653592 \qquad \text{(correct to 11 places)}$$

जगदुर्बुधाः – This is what the wise people say. Who are these wise people whom Mādhava is referring to?

13 *Vibudha*=33, *Netra*=2, *Gaja*=8, *Ahi*=8, *Hutāśana*=3, *Triguṇa*=3, *Veda*=4, *Bha*=27, *Vāraṇa*=8, *Bāhu*=2, *Nava-nikharva*=9×10^{11}. (The word *nikharva* represents 10^{11}).

Infinite series for π – as given in *Yukti-dīpikā*

व्यासे वारिधिनिहते रूपहृते व्याससागराभिहते।
त्रिशरादि विषमसङ्ख्याभक्तम् ऋणं स्वं पृथक् क्रमात् कुर्यात्॥

The diameter multiplied by four and divided by unity (is found and stored). Again the products of the diameter and four are divided by the odd numbers like three, five, etc., and the results are subtracted and added in order (to the earlier stored result).

- *vyāse vāridhinihate* $\rightarrow$ 4 $\times$ Diameter (*vāridhi*)
- *viṣamasaṅkhyābhaktam* $\rightarrow$ Divided by odd numbers
- *triśarādi* $\rightarrow$ 3, 5, etc. (*bhūtasaṅkhyā* system)
- *ṛṇaṃ svaṃ* $\rightarrow$ to be subtracted and added [successively]

$$Paridhi = 4 \times Vy\bar{a}sa \times \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\right)$$

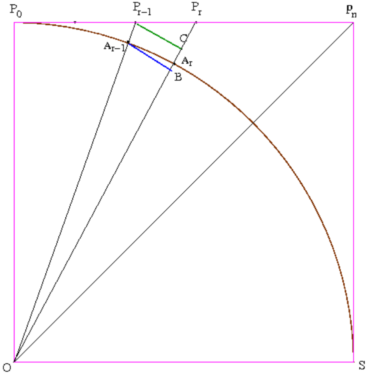
Infinite series for π

The triangles $OP_{i-1}C_i$ and $OA_{i-1}B_i$ are similar. Hence,

$$\frac{A_{i-1}B_i}{OA_{i-1}} = \frac{P_{i-1}C_i}{OP_{i-1}} \qquad (16)$$

Similarly triangles $P_{i-1}C_iP_i$ and P_0OP_i are similar. Hence,

$$\frac{P_{i-1}C_i}{P_{i-1}P_i} = \frac{OP_0}{OP_i} \qquad (17)$$



Infinite series for π

From these two relations we have,

$$\begin{aligned} A_{i-1}B_i &= \frac{OA_{i-1}.OP_0.P_{i-1}P_i}{OP_{i-1}.OP_i} \\ &= P_{i-1}P_i \times \frac{OA_{i-1}}{OP_{i-1}} \times \frac{OP_0}{OP_i} \\ &= \left(\frac{r}{n}\right) \times \frac{r}{k_{i-1}} \times \frac{r}{k_i} \\ &= \left(\frac{r}{n}\right)\left(\frac{r^2}{k_{i-1}k_i}\right). \end{aligned} \tag{18}$$

It is $\left(\frac{r}{n}\right)$ that is refered to as *khaṇḍa* in the text. The text also notes that, when the *khaṇḍa*-s become small (or equivalently n becomes large), the Rsines can be taken as the arc-bits itself.

परिधिखण्डस्यार्धज्या $\rightarrow$ परिध्यंश

i.e., $A_{i-1}B_i \rightarrow A_{i-1}A_i$.

(local approximation by linear functions i.e., tangents/differentiation)

Infinite series for π (Error estimate)

Though the value of $\frac{1}{8}$th of the circumference has been obtained as

$$\frac{C}{8} = \left(\frac{r}{n}\right)\left[\left(\frac{r^2}{k_0 k_1}\right) + \left(\frac{r^2}{k_1 k_2}\right) + \left(\frac{r^2}{k_2 k_3}\right) + \cdots + \left(\frac{r^2}{k_{n-1} k_n}\right)\right], \quad (19)$$

there may not be much difference in approximating it by either of the following expressions:

$$\frac{C}{8} = \left(\frac{r}{n}\right)\left[\left(\frac{r^2}{k_0^2}\right) + \left(\frac{r^2}{k_1^2}\right) + \left(\frac{r^2}{k_2^2}\right) + \cdots + \left(\frac{r^2}{k_{n-1}^2}\right)\right] \quad (20)$$

$$or \quad \frac{C}{8} = \left(\frac{r}{n}\right)\left[\left(\frac{r^2}{k_1^2}\right) + \left(\frac{r^2}{k_2^2}\right) + \left(\frac{r^2}{k_3^2}\right) + \cdots + \left(\frac{r^2}{k_n^2}\right)\right] \quad (21)$$

The difference between (21) and (20) will be

$$\begin{aligned}
\left(\frac{r}{n}\right)\left[\left(\frac{r^2}{k_0^2}\right) - \left(\frac{r^2}{k_n^2}\right)\right] &= \left(\frac{r}{n}\right)\left[1 - \left(\frac{1}{2}\right)\right] \qquad (\, k_0^2, k_n^2 = r^2, 2r^2) \\
&= \left(\frac{r}{n}\right)\left(\frac{1}{2}\right) \qquad (22)
\end{aligned}$$

खण्डस्य अल्पत्ववशात् तदन्तरं शून्यप्रायमेव।

Infinite series for π

Thus we have,

$$
\begin{aligned}
\frac{C}{8} &= \sum_{i=1}^{n} \frac{r}{n}\left(\frac{r^2}{k_i^2}\right) \qquad \textit{summming up/integration} \\
&= \sum_{i=1}^{n}\left[\frac{r}{n} - \frac{r}{n}\left(\frac{k_i^2 - r^2}{r^2}\right) + \frac{r}{n}\left(\frac{k_i^2 - r^2}{r^2}\right)^2 - \ldots\right] \\
&= \left(\frac{r}{n}\right)[1 + 1 + \ldots + 1] \\
&\quad - \left(\frac{r}{n}\right)\left(\frac{1}{r^2}\right)\left[\left(\frac{r}{n}\right)^2 + \left(\frac{2r}{n}\right)^2 + \ldots + \left(\frac{nr}{n}\right)^2\right] \\
&\quad + \left(\frac{r}{n}\right)\left(\frac{1}{r^4}\right)\left[\left(\frac{r}{n}\right)^4 + \left(\frac{2r}{n}\right)^4 + \ldots + \left(\frac{nr}{n}\right)^4\right] \\
&\quad - \left(\frac{r}{n}\right)\left(\frac{1}{r^6}\right)\left[\left(\frac{r}{n}\right)^6 + \left(\frac{2r}{n}\right)^6 + \ldots + \left(\frac{nr}{n}\right)^6\right] \\
&\quad + \ldots \quad .
\end{aligned} \tag{23}
$$

Infinite series for π

If we take out the powers of *bhujā-khaṇḍa* $\frac{r}{n}$, the summations involved are that of even powers of the natural numbers, namely *edādyekottara-varga-saṅkalita*, $1^2 + 2^2 + ... + n^2$, *edādyekottara-varga-varga-saṅkalita*, $1^4 + 2^4 + ... + n^4$, and so on. Kerala astronomers knew that

$$\sum_{i=1}^{n} i^k \approx \frac{n^{k+1}}{k+1}. \tag{24}$$

Thus, we arrive at the result

$$\frac{C}{8} = r\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right), \tag{25}$$

which is given in the form

$$Paridhi = 4 \times Vy\bar{a}sa \times \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\cdots\right)$$

The Cons

Do the popular books present the right history?

Recently I met Alex Bellos, a British journalist, who came interview a few of the historians of mathematics in India. Having interviewed, before departing he handed a over a book to me authored in 2010.

For two thousand years the only way to pinpoint pi was by using polygons. But, in the seventeenth century, Gottfried Leibniz and John Gregory ushered in a new age of pi appreciation with the formula:

$$\frac{pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \dots$$

// Historically incorrect

In other words, a quarter of pi is equal to one minus a third plus a fifth minus a seventh plus a ninth and so on, alternating the addition and subtraction of unit fractions of the odd numbers as they head to infinity. Before this point scientists were aware only of the scattergun randomness of pi's decimal expansion. Yet here was one of the most elegant and

Concluding Remarks

What does a formal mathematical proof constitute?

- According to Hilbert[14]:

  > *A mathematical proof consists of a finite sequence of statements, each of which is an axiom or derived from preceding axioms by the use of modus ponens or similar rules of reasoning. . . . being a sequence of statements, a reference to empirical cannot be introduced in the course of a proof.*

- Even axioms are not regarded as self-evident truths; they are merely arbitrary set of propositions whose necessary consequences are explored in the mathematical theory.
- As they are some form of 'tautologies' they are not refutable.
- Postulates related to the empirical world lead to physical theory and not mathematical theory.

[14] Paraphrased by C. K. Raju (p. 62) in his Cultural Foundations of Mathematics.

Concluding Remarks

Concept of upapatti or vāsanā in Indian tradition

- The *upapatti*-s of Indian mathematics, do not form a part of logical deductions based on a set of axioms. This nevertheless does not mean that *upapatti* would be illogical.
- The arguments presented in *upapatti* need not always be universal in nature, i.e., they could be context specific.
- Etymologically the word *upapatti* can be derived from the root (*dhātu*) "*pad*", which means "understanding", with a prefix and suffix added to it.

  upa + *pad* + *ktin* = *upapatti*.
  That which brings understanding closer.

- The term *vāsanā* is also synonymously employed in the place of *upapatti*, which means "to dwell/reside". With this in mind, the derivation of the word *vāsanā* can be shown as follows:

  That which makes the enunciated principle reside/stay [deeply] in the minds of the reader.[15]

- Thus, *upapatti* or *vāsanā* is something that makes you wiser.

Concluding Remarks

- The episode of C. M. Whish was quite revealing to know the that there was a lot of hesitation in accepting that the series for π—in its different avatars—could have been invented by Hindus.[16]
- Besides arriving at the series, the analysis that has gone in—with absolute logical rigor—to obtain several rapidly convergent series is indeed remarkable.
- Why were they worried about very accurate values of π ?
- Accuracy of π $\rightarrow$ Accuracy of $Trijy\bar{a}\ R$ $\rightarrow$ Accuracy in the computation of sines $\rightarrow$ Accuracy in planetary positions $\rightarrow$ Accuracy in the determination of tithis, and so on, $\rightarrow$ Avoid incompleteness.[17]
- Perhaps it is the philosophical difference, that made the historians declare: no evidence that they (Hindus) considered proof at all — that hardly has any truth value!

[16] I guess it would have been a big challenge for Charles Whish to get across the idea that the series has been invented by the Natives.

[17] नास्ते कालावयवकलना …श्रौतस्मार्तव्यवहृतिरपि छिद्यते तत्र धर्माः।

Excerpts from David Mumford's Preface / Review

David Mumford[18] observes:

> *Only a fraction of this has become generally known to mathematicians in the West. Too many people still think that mathematics was born in Greece and more or less slumbered until the Renaissance.*

It is right time that the full story of Indian mathematics from Vedic times through 1600 became generally known. I am not miniminzing the genius of the Greeks and their wonderful invention of pure mathematics, but other people have been doing math in different ways and they have often attained the same goals independently. Rigorous mathematics in the Greek style should not be seen as the only way to gain mathematical knowledge . . . .

The muse of mathematics can be wooed in many different ways and her secrets teased out of her.

[18] A renowned mathematician and Fields medalist.

Concluding Remarks

History vs. Myth-making

Finally, again, I would like to conclude with the words of Claude Alvares[19] –

- All History is elaborate efforts in myth-making. ...
- If we must continue to live with myths, however, it is far better we choose to live with those of our own making rather than by those invented by others for their own purposes.
- That much at least we owe as an 'independent' thinkers and researchers !!.

[19] In his introduction to *The Indian Science and Technology in the 18th Century*, Other India Press, Goa.

Thanks!

॥ धन्यवादाः ॥