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# Weihrauch complexity and the Hagen school of computable analysis

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## Abstract

Weihrauch complexity is now an established and active part of mathematical logic. It can be seen as a computability-theoretic approach to classifying the uniform computational content of mathematical problems. This theory has become an important interface between more proof-theoretic and more computability-theoretic studies in the realm of reverse mathematics. Here we present a historical account of the early developments of Weihrauch complexity by the Hagen school of computable analysis that started more than thirty years ago, and we indicate how this has influenced, informed, and anticipated more recent developments of the subject.

## 1 Computable analysis in Hagen

The Hagen school of computable analysis was founded by Klaus Weihrauch in the 1980s. Weihrauch received his PhD from the University of Bonn in 1973, and after a research associateship at Cornell University and a professorship position at RWTH Aachen he held a chair for Theoretical Computer Science at the University of Hagen from 1979 until his retirement in 2008. Together with his PhD students he developed the representation based approach to computable analysis, sometimes called *type-2 theory of effectivity*. Most notably among the early PhD students were Christoph Kreitz [68], Thomas Deil [30] and Norbert Müller [78]. A later generation of PhD students includes Peter Hertling [48], Xizhong Zheng [113], Matthias Schröder [95], the author of these notes, and others.<sup>1</sup>

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<sup>1</sup>A more complete Ph.D. genealogy of Weihrauch can be found in the preface of [21].

The main idea of this approach to computable analysis is to perform all computability considerations on Baire space, and to transfer these concepts to other spaces by representing them with Baire space. On the one hand, on Baire space  $\mathbb{N}^{\mathbb{N}}$  concepts such as continuity and computability are well understood, for instance with the help of Turing machines that operate on natural number sequences. On the other hand, natural number sequences can naturally be used to represent other objects such as real numbers, closed subsets or continuous functions on real numbers. Analogous statements hold for Cantor space  $2^{\mathbb{N}}$ , which additionally has natural concepts of time and space complexity.

Implicitly, such representations were used ever since Turing [102, 101] introduced his machines to operate on real numbers, and they are also implicit in constructive analysis [4], reverse mathematics [98], descriptive set theory [76, 63] and in set-theoretical constructions in classical mathematics too. However, the crucial idea of Weihrauch and his collaborators was to make such representations objects of mathematical investigations themselves by considering them as partial surjective maps.

**Definition 1.1** (Representation). A *representation* of a set  $X$  is a partial surjective map  $\delta : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow X$ .<sup>2</sup>

This idea already had a tradition in mathematical logic since it was also present in Hauck’s work [43, 44]. Kreitz and Weihrauch [70, 107, 71, 112] developed the theory of representations following the lines of the theory of numberings that was proposed in the 1970s by Ershov [35]. An early manifesto by Kreitz and Weihrauch can be found in [69] and more complete presentations in [108, 111].

It turned out that when dealing with infinite objects such as real numbers the proper choice of a representation is more crucial than for discrete objects. When one represents rational numbers  $\mathbb{Q}$  one more or less automatically arrives at a suitable representation from the perspective of computability, and one has to work hard to find a representation that is not computably equivalent to a natural one, for instance by artificially encoding the halting problem into the representation.

In the case of infinite objects, such as real numbers  $\mathbb{R}$ , there are many natural representations that lead to mutually inequivalent structures. Figure 1 displays a portion of the lattice of real number representations that were studied already by Deil [30]. We omit the formal definitions, but we introduce some symbolic names in the diagram for later reference. The representations of the same shade of grey induce identical notions of computable real numbers: the Cauchy representation and all representations

<sup>2</sup>In this paper, the notation  $f : \subseteq X \rightarrow Y$  abbreviates “ $f$  is a function with  $\text{dom}(f) \subseteq X$  and  $\text{ran}(f) \subseteq Y$ ”. Similarly,  $f : \subseteq X \rightrightarrows Y$  abbreviates “ $f$  is a multi-valued function with  $\text{dom}(f) \subseteq X$  and  $\text{ran}(f) \subseteq Y$ ”, i.e.,  $f \subseteq X \times Y$ .

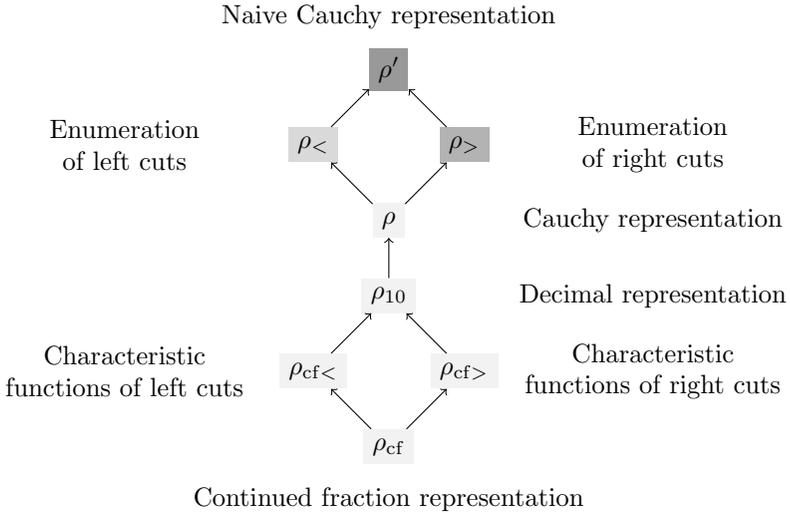


FIGURE 1. Lattice of real number representations.

below it in the diagram induce the ordinary notion of a computable real number; the representations via enumerations of left and right cuts induce the left- and right-computable real numbers, respectively; the naïve Cauchy representation induces the limit computable reals.

The preorder used in the diagram is that of reducibility of representations, which can be seen as a generalization of the concept of reducibility for numberings or of many-one reducibility to type-2 objects. Any arrow in the diagram indicates a reduction in the direction of the arrow, and all missing arrows indicate that no reduction is possible (except for those that follow by transitivity and reflexivity).

**Definition 1.2** (Reducibility). Let  $f, g : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow X$  be partial functions. Then we say that  $f$  is reducible to  $g$ , in symbols  $f \leq g$ , if there exists a computable  $F : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  such that  $f(p) = g \circ F(p)$  for all  $p \in \text{dom}(f)$ .

By  $\equiv$  we denote the equivalence induced by  $\leq$ . One can also define a purely topological analogue of this reducibility by requiring that  $F$  is continuous, and it turns out that the relations in the diagram in Figure 1 are not affected by this modification. Hence, one can say that the uniform distinctions between these representations are already of topological nature.

In particular, the continued fraction representation carries more continuously accessible information about real numbers than the decimal representation, and in turn the decimal representation carries more information

$$\begin{array}{ccc}
 \mathbb{N}^{\mathbb{N}} & \xrightarrow{F} & \mathbb{N}^{\mathbb{N}} \\
 \delta_X \downarrow & & \downarrow \delta_Y \\
 X & \xrightarrow{f} & Y
 \end{array}$$

FIGURE 2. Continuity with respect to representations.

than the Cauchy representation. Having more informative representations on the input side is helpful, but it can be a burden on the output side. For instance addition is neither computable with respect to the continued fraction representation [66] nor with respect to the decimal representation [101].

These observations naturally lead to the question how one can identify a suitable representation among all the many representations of infinite objects that one can consider? The answer that Kreitz and Weihrauch gave is that a good representation has to be topologically natural, and the corresponding concept is called *admissibility*. Their main theorem on this topic is the following [69, 70, 108].

**Theorem 1.3** (Kreitz-Weihrauch 1984). *If  $X$  and  $Y$  are admissibly represented  $T_0$ -spaces with countable bases, then  $f : \subseteq X \rightarrow Y$  is continuous if and only if it is continuous with respect to the underlying representations.*

Some further explanations are required here. For one, continuity with respect to the underlying representations means that the diagram in Figure 2 commutes.

That is, if  $(X, \delta_X)$  and  $(Y, \delta_Y)$  are represented spaces, then a function  $f : \subseteq X \rightarrow Y$  is called *continuous with respect to the underlying representations*, if there is a continuous  $F : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  such that  $\delta_Y F(p) = f\delta_X(p)$  for all  $p \in \text{dom}(f\delta_X)$ . Other notions such as computability, Borel measurability, etc., can be transferred analogously to represented spaces, and Theorem 1.3 tells us that as long we use admissible representations, then at least topological continuity is the same as continuity with respect to the representations.

Theorem 1.3 was later generalized by Schröder [95] to a larger category of topological spaces, and one needs to replace topological continuity with sequential continuity in this more general context. Schröder also introduced a more general definition of admissibility that we are going to present here.

**Definition 1.4** (Admissibility). A representation  $\delta : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow X$  of a topological space  $X$  is called *admissible* if it is continuous and maximal among all continuous representations of  $X$  with respect to the topological version of the reducibility  $\leq$ .

Among the real number representations it is the equivalence class of the Cauchy representation that yields admissible representations with respect to the Euclidean topology. In general, admissibility yields a handy criterion to judge whether a representation is suitable from a topological perspective. Many hyper and function space representations have been analyzed, such as representations for the space  $\mathcal{C}(X)$  of continuous functions  $f : X \rightarrow \mathbb{R}$ , which we are not going to introduce in detail here [22, 111]. There are also other schools of computable analysis that approach the subject from a slightly different angle, such as the Pour-El and Richards school [92]. Studies of computational complexity in analysis were initiated by Norbert Müller [77, 79, 80], Ker-I Ko [67], and more recently by Akitoshi Kawamura and Stephen Cook [61, 62]. A more comprehensive discussion of the historical developments in computable analysis that arose out of Turing's work is presented in [2]. The tutorial [22] contains a concise introduction to computable analysis, and the handbook [20] contains surveys on many fascinating aspects of the more recent developments in computable analysis in general. From 1995 until today the computable analysis group founded in Hagen runs a conference with the title *Computability and Complexity in Analysis* (CCA). Some participants of the first meeting CCA 1995 in Hagen are shown in Figure 3.

## 2 Weihrauch reducibility

When studies in computable analysis progressed, the need arose to include the study of multi-valued maps. For instance, when one discusses the problem of finding zeros of a continuous function  $f : X \rightarrow \mathbb{R}$ , then it is not sufficient to consider single-valued functions of type  $Z : \subseteq \mathcal{C}(X) \rightarrow \mathbb{R}$  in order to describe zero finding, because under certain assumptions it might be possible to compute a zero of a continuous function  $f : X \rightarrow \mathbb{R}$  only in a non-extensional way, i.e., such that the zero depends on the description of  $f$  and not just on  $f$  itself [111]. This phenomenon is best captured by describing zero finding as a multi-valued partial function of type  $Z : \subseteq \mathcal{C}(X) \rightrightarrows \mathbb{R}$  (cf. Footnote 2). In general, many mathematical problems can be construed as such multi-valued maps. Hence we use the following general definition.

**Definition 2.1** (Problem). A *problem*  $f : \subseteq X \rightrightarrows Y$  is a partial multi-valued map on represented spaces  $X, Y$ .

If we have two problems  $f, g : \subseteq X \rightrightarrows Y$  of the same type, then we say that  $f$  *refines*  $g$ , in symbols  $f \sqsubseteq g$ , if  $\text{dom}(g) \subseteq \text{dom}(f)$  and  $f(x) \subseteq g(x)$



FIGURE 3. A group picture with some participants of CCA 1995. Names from left to right: Klaus Weihrauch, Norbert Müller, Ludwig Staiger, Kostas Skandalis, Markus Bläser, Ker-I Ko, Peter Hertling, Jens Blanck, Matthias Schröder, Arthur Chou, Klaus Meer, Martin Hötzel Escardo, Pietro Di Gianantonio, Uwe Mylatz, Holger Schulz, Rudolf Freund, Janos Blazi, Achim Kallweit. Picture: Vasco Brattka.

for all  $x \in \text{dom}(g)$ . Problems can naturally be combined in several ways. For instance, the *composition*  $g \circ f$  of two problems  $f : \subseteq X \rightrightarrows Y$  and  $g : \subseteq Y \rightrightarrows Z$  is defined by  $\text{dom}(g \circ f) := \{x \in \text{dom}(f) : f(x) \subseteq \text{dom}(g)\}$  and

$$(g \circ f)(x) := \{z \in Z : (\exists y \in f(x)) z \in g(y)\}$$

for all  $x \in \text{dom}(g \circ f)$ . The particular choice of the domain is important, as this ensures that composition preserves computability and continuity in the appropriate way. Another operation on the problems  $f : \subseteq X \rightrightarrows Y$  and  $g : \subseteq W \rightrightarrows Z$  is the *product*  $f \times g$  that is defined by  $\text{dom}(f \times g) := \text{dom}(f) \times \text{dom}(g)$  and

$$(f \times g)(x, w) := f(x) \times g(w)$$

for all  $(x, w) \in \text{dom}(f \times g)$ .

As soon as problems are captured as multi-valued maps, there arises the need for a tool to compare the computational power of problems beyond refinement. In some sense, the concept of reducibility given in Definition 1.2 already gives us a way to compare (single-valued) problems of a certain type. However, only the input is subject to a pre-processing step here. Weihrauch's ideas of extending the notion of many-one reducibility such that also a post-processing of the output is considered, were laid out in two unpublished technical reports [109, 110]:

Klaus Weihrauch, *The degrees of discontinuity of some translators between representations of the real numbers*, Technical Report TR-92-050, International Computer Science Institute, Berkeley, July 1992.

Klaus Weihrauch, *The TTE-interpretation of three hierarchies of omniscience principles*, Informatik Berichte 130, FernUniversität Hagen, September 1992.

In his original definition, Weihrauch did not present the definition of his reducibilities in the way it is seen most often now. In order to make the definition as clear and simple as possible, we present a slightly more general definition in almost categorical terms that can be interpreted in different ways. By  $\text{id} : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  we denote the identity on Baire space.

**Definition 2.2** (Weihrauch reducibility). Let  $C$  be a class of problems, and let  $f, g$  be two problems. We introduce the following terminology:

1.  $f$  is *strongly Weihrauch reducible* to  $g$  (with respect to  $C$ ), in symbols  $f \leq_{\text{sw}} g$ , if there are  $H, K \in C$  such that  $H \circ g \circ K \sqsubseteq f$ .
2.  $f$  is *Weihrauch reducible* to  $g$  (with respect to  $C$ ), in symbols  $f \leq_{\text{w}} g$ , if there are  $H, K \in C$  such that  $H \circ (\text{id} \times g) \circ K \sqsubseteq f$ .

The set  $C$  has to be fixed in order to make the symbolic notation meaningful.

In the usual definition, we use for  $C$  the class of all computable (or continuous) problems.<sup>3</sup> And this is what usually is called (the topological version of) Weihrauch reducibility. If  $C$  is sufficiently nice, i.e., closed under composition and under product with  $\text{id}$ , then the above definitions actually yield preorders, no matter what  $C$  is. And in fact, versions of Weihrauch reducibility have also been considered for the classes  $C$  of (suitably defined) polynomial-time computable problems [62], arithmetic problems [75], or even hyperarithmetic piecewise computable problems [42]. In a more categorical setting concepts similar to Weihrauch reducibility were also studied by Hirsch [55], and in complexity theory similar concepts are known as (polynomial-time) many-one reducibility for functions on discrete spaces [62].

The intuitive idea of (strong) Weihrauch reducibility is illustrated in the diagrams in Figure 4. The idea is that  $f$  being *strongly Weihrauch reducible* to  $g$  means that  $g$  composed with some suitable pre-processor  $K$  and some suitable post-processor  $H$  refines  $f$ . Suitability includes that the pre- and post-processors have to be from the set  $C$ . In the case of the ordinary (non-strong) reduction the post-processor additionally has access to the original input in form of some information determined by the pre-processor.

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<sup>3</sup>That this actually yields the ordinary definition of Weihrauch reducibility follows from [25, Lemma 2.5].

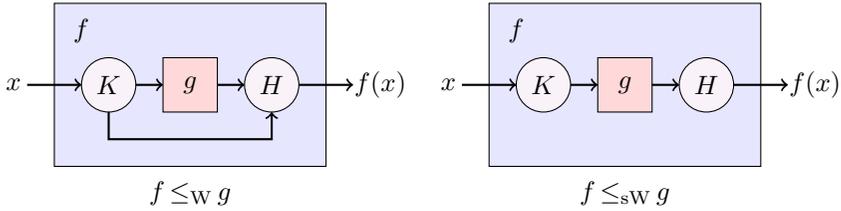


FIGURE 4. Weihrauch reducibility and strong Weihrauch reducibility.

In the original definition in [109, 110], Weihrauch firstly only considered single-valued problems  $f, g$  on Cantor space, and for  $C$  he used the set of continuous single-valued functions on Cantor space. He denoted the reducibilities  $\leq_{sW}$  and  $\leq_W$  by  $\leq_2$  and  $\leq_1$ , respectively (and he used  $\leq_0$  for a many-one like reducibility as in Definition 1.2). In a second step in [109, 110] he extended the definition to sets of such problems  $f, g$ . But this is just another technical way of dealing with multi-valuedness and essentially yields an approach that is equivalent to the modern one.<sup>4</sup> Hence, if not mentioned otherwise, we will assume from now on that  $C$  is the class of all computable problems, and we will express everything in modern terminology. As usual, we denote the equivalences derived from strong and ordinary Weihrauch reducibility by  $\equiv_{sW}$  and  $\equiv_W$ , respectively, and the strict versions of the reducibilities by  $<_{sW}$  and  $<_W$ , respectively.

Only relatively late, it was discovered that the order structures induced by strong and ordinary Weihrauch reducibility are lattices. For  $\leq_W$  this result is due to the author and Gherardi [13] (for the lower semi-lattice) and Pauly [90] (for the upper semi-lattice). For  $\leq_{sW}$  the result is due to Dzhafarov [32].

**Proposition 2.3** (Weihrauch lattice). The order structures induced by  $\leq_W$  and  $\leq_{sW}$  are lattices. In the case of  $\leq_W$  the lattice is distributive, in the case of  $\leq_{sW}$  it is not.

More precise definitions along these lines and further results can be found in [17].

### 3 Weihrauch complexity

One goal of Weihrauch complexity is to classify the computational content of mathematical theorems of the logical form

$$(\forall x \in X)(x \in D \implies (\exists y \in Y) P(x, y))$$

<sup>4</sup>See the discussions in [25, Section 2.1] and [31, Appendix A].

Weihrauch complexity	reverse mathematics
$C_1$	$RCA_0$ without $\Sigma_1^0$ -induction
$C_{\mathbb{N}}$	$\Sigma_1^0$ -induction
$C_{2^{\mathbb{N}}}$	$WKL_0$ without $\Sigma_1^0$ -induction
$C_{\mathbb{R}}$	$WKL_0$ with $\Sigma_1^0$ -induction
$C_{\mathbb{N}^{\mathbb{N}}}$	$ATR_0$

FIGURE 5. Choice problems versus axiom systems.

in the Weihrauch lattice. If  $X$  and  $Y$  are represented spaces, then such a theorem directly translates into a Skolem-like problem

$$F : \subseteq X \rightrightarrows Y, x \mapsto \{y \in Y : P(x, y)\}$$

with  $\text{dom}(F) = D$ . To locate the problem  $F$  in the Weihrauch lattice amounts to classifying the uniform computational complexity of the corresponding theorem. One way to calibrate the complexity of some  $F$  is to compare it to suitable *choice problems* for a suitable space  $X$ . By  $C_X$  we denoted the *closed choice problem*

$$C_X : \subseteq \mathcal{A}_-(X) \rightrightarrows X, A \mapsto A$$

for a computable metric space  $X$ . The instances are non-empty closed sets  $A \subseteq X$  represented by negative information (the space of all such closed sets is denoted by  $\mathcal{A}_-(X)$ ) and the solution can be any point in  $A$  (see [12, 17] for more precise definitions). Equivalently, we can see the instances as continuous functions  $f : X \rightarrow \mathbb{R}$  with zeros and the solution can be any zero of the function  $f$ , i.e., if  $\mathcal{C}(X)$  denotes the space of continuous function  $f : X \rightarrow \mathbb{R}$ , represented in a natural way, then

$$C_X : \subseteq \mathcal{C}(X) \rightrightarrows X, f \mapsto f^{-1}\{0\},$$

with  $\text{dom}(C_X) = \{f \in \mathcal{C}(X) : f^{-1}\{0\} \neq \emptyset\}$ . This alternative description of choice shows that choice problems are basically about the solutions of equations of type

$$f(x) = 0$$

for continuous  $f : X \rightarrow \mathbb{R}$ . Typical spaces  $X$  whose choice problems have been considered are Cantor space  $X = 2^{\mathbb{N}}$ , Baire space  $X = \mathbb{N}^{\mathbb{N}}$ , Euclidean space  $X = \mathbb{R}$ , the natural numbers  $X = \mathbb{N}$ , and finite spaces  $n = \{0, \dots, n - 1\}$  for  $n \in \mathbb{N}$ .

Proving equivalences to choice problems roughly corresponds to a uniform version of a classification of the corresponding theorem in reverse

mathematics [98]. Reverse mathematics is a proof-theoretic approach to classifying theorems according to which axioms are needed to prove the theorem in second-order arithmetic. Typical axiom systems are *recursive comprehension*  $\text{RCA}_0$ , *arithmetic comprehension*  $\text{ACA}_0$ , *arithmetic transitive recursion*  $\text{ATR}_0$ . Details can be found in [98, 56]. The table given in Figure 5 indicates the correspondences between Weihrauch complexity and reverse mathematics.

This is just a very rough correspondence. For instance,  $\text{ATR}_0$  can be analyzed more closely in the vicinity of  $\mathbf{C}_{\mathbb{N}^{\mathbb{N}}}$ , and this is subject of several recent studies, for instance, by Marcone and Valenti [75], Goh, Pauly, and Valenti [41], Goh [40], Kihara, Marcone and Pauly [64].

The system  $\text{ACA}_0$  can be characterized using iterations of the *limit problem*

$$\text{lim} : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}, \langle p_0, p_1, p_2, \dots \rangle \rightarrow \lim_{n \rightarrow \infty} p_n,$$

which is just the usual limit on Baire space (where for technical convenience, the input sequence is encoded by a standard tupling function  $\langle \rangle$  in a single point in Baire space). Using these choice problems, one can obtain the following results. We are just mentioning some example for  $\mathbf{C}_{\mathbb{N}}$ ,  $\mathbf{C}_{2^{\mathbb{N}}}$  and  $\text{lim}$ . Many further results can be found in the survey [17]. The following result on problems equivalent to  $\mathbf{C}_{\mathbb{N}}$  is due to the author and Gherardi [13].

**Theorem 3.1** (Choice on the natural numbers). The following are all Weihrauch equivalent to each other:

1. Choice on natural numbers  $\mathbf{C}_{\mathbb{N}}$ .
2. The Baire category theorem for computable complete metric spaces.
3. Banach's inverse mapping theorem for the Hilbert space  $\ell_2$ .
4. The open mapping theorem for  $\ell_2$ .
5. The closed graph theorem for  $\ell_2$ .
6. The uniform boundedness theorem on non-singleton computable Banach spaces.

In order to be more precise, one would have to explain which logical version of the theorem is translated into a problem here, and how the underlying data are represented. That these aspects can make a difference has been discussed for the Baire category theorem in detail [19]. We will not formalize these problems here and refer the interested reader to [13] for all details.

Next we mention a number of problems that are equivalent to  $\mathbf{C}_{2^{\mathbb{N}}}$ . The result on the Hahn-Banach theorem and the separation theorem is due to

Gherardi and Marcone [38], the result on the Brouwer fixed point theorem is due to the author, Le Roux, Joseph Miller, and Pauly [23]. The result on the Gale-Stewart theorem is due to Le Roux and Pauly [74]. All other results are easy to prove, the Heine-Borel theorem and the theorem of the maximum were briefly discussed in [8].

**Theorem 3.2** (Choice on Cantor space). The following are all Weihrauch equivalent to each other:

1. Choice on Cantor space  $C_{2^{\mathbb{N}}}$ .
2. Weak König's lemma WKL.
3. The Hahn-Banach theorem.
4. The separation theorem SEP on the separation of two disjoint enumerated sets in  $\mathbb{N}$  by the characteristic function of another set (cf. below).
5. The Heine-Borel covering theorem.
6. The theorem of the maximum.
7. The Brouwer fixed point theorem for dimension  $n \geq 2$ .
8. The theorem of Gale-Stewart (on determinacy of games on Cantor space with closed winning sets).

Once again one would have to make these problems more precise. We give two examples. The *separation problem* is defined by

$$\text{SEP} \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows 2^{\mathbb{N}}, \langle p, q \rangle \mapsto \{A \in 2^{\mathbb{N}} : \text{ran}(p-1) \subseteq A \subseteq \mathbb{N} \setminus \text{ran}(q-1)\}$$

with  $\text{dom}(\text{SEP}) := \{\langle p, q \rangle : \text{ran}(p-1) \cap \text{ran}(q-1) = \emptyset\}$ . Here  $p-1 \in \mathbb{N}^* \cup \mathbb{N}^{\mathbb{N}}$  is the finite or infinite sequence of natural numbers that consists of the concatenation of

$$p(0) - 1, p(1) - 1, p(2) - 1, \dots$$

with the understanding that  $-1$  is the empty word. This technical construction is used to allow for enumerations of the empty set, and in order to keep 0 as dummy value in enumerations. We are going to see it later again.

Weak König's lemma is defined by

$$\text{WKL} \subseteq \text{Tr} \rightrightarrows 2^{\mathbb{N}}, T \mapsto [T],$$

where  $\text{Tr}$  is the set of binary trees,  $\text{dom}(\text{WKL})$  contains all infinite such trees, and  $[T]$  is the set of infinite paths of such a tree  $T$ . As a simple example we mention at least one proof [13, Proposition 2.8 and Theorem 2.11]. If we

represent the space  $\mathcal{A}_-(2^{\mathbb{N}})$  of closed subsets of  $2^{\mathbb{N}}$  by negative information, then the map  $f : \text{Tr} \rightarrow \mathcal{A}_-(2^{\mathbb{N}}), T \rightarrow [T]$  is easily seen to be computable and it admits a multi-valued computable right inverse  $g : \mathcal{A}_-(2^{\mathbb{N}}) \rightrightarrows \text{Tr}$ . This is all that is needed to prove  $\text{WKL} \equiv_{\text{W}} \text{C}_{2^{\mathbb{N}}}$ .

We close this section with a number of problems that are equivalent to the limit operation. The results on the monotone convergence theorem and on the operator of differentiation  $d$  (which is restricted to continuously differentiable functions) are due to von Stein [106] (see also Theorem 6.1). The Radon-Nikodym theorem was studied by Hoyrup, Rojas, and Weihrauch [58]. The other results are easy to show (see for instance [12, 8, 9]).

**Theorem 3.3** (The limit). The following are all Weihrauch equivalent to each other:

1. The limit map  $\text{lim}$  on Baire space (or Cantor space, or Euclidean space).
2. The Turing jump  $J : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}, p \mapsto p'$ .
3. The monotone convergence theorem  $\text{MCT} : \subseteq \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R}, (x_n)_{n \in \mathbb{N}} \mapsto \sup_{n \in \mathbb{N}} x_n$ .
4. The operator of differentiation  $d : \subseteq \mathcal{C}[0, 1] \rightarrow \mathcal{C}[0, 1], f \mapsto f'$ .
5. The Fréchet-Riesz representation theorem for  $\ell_2$ .
6. The Radon-Nikodym theorem.

In between the degrees of  $\text{C}_{2^{\mathbb{N}}}$  and  $\text{lim}$  there is the degree of  $\text{C}_{\mathbb{R}}$  and the degree of the *lowness problem*  $\text{L} := J^{-1} \circ \text{lim}$ . The following uniform version of the low basis theorem of Jockusch and Soare [60] was proved by the author, de Brecht and Pauly [12].

**Theorem 3.4** (Uniform low basis theorem).  $\text{C}_{2^{\mathbb{N}}} <_{\text{W}} \text{C}_{\mathbb{R}} <_{\text{W}} \text{L} <_{\text{W}} \text{lim}$ .

This implies, in particular,  $\text{C}_{\mathbb{N}} <_{\text{W}} \text{L}$ . There are many further studies, e.g., on Nash equilibria by Arno Pauly [89, 91] that can also be described with the help of choice, on probabilistic versions of choice, for instance, by the author and Pauly [24], the author, Gherardi and Hölzl [15], Bienvenu and Kuyper [3], on degrees defined by jumps of choice, e.g., by the author, Gherardi and Marcone [16], non-standard degrees obtained by Ramsey's theorem, by Dorais, Dzhafarov, Hirst, Mileti and Shafer [31], Hirschfeldt and Jockusch [56, 57], the author and Rakotoniaina [26], Dzhafarov, Goh, Hirschfeldt, Patey and Pauly [33], Cholak, Dzhafarov, Hirschfeldt and Patey [28], Marcone and Valenti [75], and Dzhafarov and Patey [34].

The classifications presented here are essentially in line with results from reverse mathematics [98]. There are several attempts to establish formal bridges between the proof-theoretic side of reverse mathematics to the more uniform computational side of Weihrauch complexity. These can be found, for instance, in work of Fujiwara [36], Uftring [104], and Kuyper [72].

## 4 Degrees of translators between real number representations

We are now going to discuss some of Weihrauch's early results from [109, 110]. We will try to put them into the context of more recent results with the aim to show that many important Weihrauch degrees were already anticipated in these initial reports. The main objective of [109] is the study of a kind of quotient structure of the diagram displayed in Figure 1. For any two representations  $\delta_1, \delta_2$  of some set  $X$  we can consider the *implication problem*  $(\delta_1 \rightarrow \delta_2) : \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$

$$(\delta_1 \rightarrow \delta_2)(p) := \{q \in \mathbb{N}^{\mathbb{N}} : \delta_2(q) = \delta_1(p)\}$$

with  $\text{dom}(\delta_1 \rightarrow \delta_2) := \text{dom}(\delta_1)$ , which measures the complexity of translating  $\delta_1$  into  $\delta_2$ . In particular,  $(\delta_1 \rightarrow \delta_2)$  is computable (in the sense that it has a computable refinement) if and only if  $\delta_1 \leq \delta_2$ . A benchmark that can be used to measure the complexity of other problems is the problem

$$\text{EC} : \mathbb{N}^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}, p \mapsto \text{ran}(p - 1)$$

that translates an enumeration of a set into its characteristic function. Using this terminology one of the results obtained by Weihrauch is the following.

**Theorem 4.1** (Weihrauch [109, Theorem 4]). We obtain

$$\text{EC} \equiv_{\text{W}} (\rho' \rightarrow \rho) \equiv_{\text{W}} (\rho' \rightarrow \rho_{>}) \equiv_{\text{W}} (\rho_{<} \rightarrow \rho_{>}) \equiv_{\text{W}} (\rho_{<} \rightarrow \rho).$$

The proof of this proposition builds on earlier results by von Stein [106]. While Theorem 4.1 describes quotients in the upper part of the diagram in Figure 1, the next result describes quotients that refer to the lower part of the diagram. In order to capture the complexity, one needs to modify the benchmark problem EC as follows. By  $\text{EC}_1 : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$  we denote the restriction of EC to such enumerations  $p$  for which

$$|\mathbb{N} \setminus \text{ran}(p - 1)| \leq 1$$

holds, i.e., the enumerated set  $\text{ran}(p - 1)$  is either  $\mathbb{N}$  or  $\mathbb{N} \setminus \{k\}$  for some  $k \in \mathbb{N}$ .

**Theorem 4.2** (Weihrauch [109, Theorem 22]). We obtain

$$\text{EC}_1 \equiv_{\text{W}} (\rho \rightarrow \rho_{\text{cf}}) \equiv_{\text{W}} (\rho \rightarrow \rho_{\text{cf}>}) \equiv_{\text{W}} (\rho_{\text{cf}<} \rightarrow \rho_{\text{cf}>}) \equiv_{\text{W}} (\rho_{\text{cf}<} \rightarrow \rho_{\text{cf}}).$$

The proof of this result can be built on the observation that  $\text{EC}_1$  is equivalent to the rationality problem (see Proposition 7.1 below), and that relative to the rationality problem the continued fraction algorithm is computable. We mention one last result from [109] that is related to the separation problem. Similarly as with  $\text{EC}$ , the separation problem  $\text{SEP}$  has a version  $\text{SEP}_1$  restricted to the separation of sets which are almost complementary in the sense that the pairs  $\langle p, q \rangle \in \text{dom}(\text{SEP}_1)$  have to satisfy the additional condition

$$|\mathbb{N} \setminus (\text{ran}(p-1) \cup \text{ran}(q-1))| \leq 1.$$

Using this terminology, we can say something on the middle part of the diagram in Figure 1. Namely, the problem  $\text{SEP}_1$  characterizes exactly the complexity of translating the Cauchy representation into the decimal representation.

**Theorem 4.3** (Weihrauch [109, Theorem 13]).  $\text{SEP}_1 \equiv_{\text{W}} (\rho \rightarrow \rho_{10})$ .

There are further results along these lines included in [109] that we cannot all summarize here. The three Weihrauch degrees of  $\text{EC}$ ,  $\text{EC}_1$  and  $\text{SEP}_1$  have anticipated classes that have been studied much later. We mention some of these observations.

**Theorem 4.4.** We obtain:

1.  $\text{SEP} \equiv_{\text{W}} \text{WKL} \leq_{\text{W}} \text{EC} \equiv_{\text{W}} \text{lim}$ ,
2.  $\text{SEP}_1 \equiv_{\text{W}} \text{C}_{\#\leq 2} \leq_{\text{W}} \text{EC}_1 \equiv_{\text{W}} \text{SORT}$ .

The equivalence  $\text{EC} \equiv_{\text{W}} \text{lim}$  follows from [7, Proposition] (see also [14, Lemma 5.3] and [12]). The equivalence  $\text{EC}_1 \equiv_{\text{W}} \text{SORT}$  is proved in Proposition 7.1 in the appendix, where also a definition of the problem  $\text{SORT}$  is given, which was originally introduced by Pauly and Neumann [85]. As mentioned before, the equivalence  $\text{SEP} \equiv_{\text{W}} \text{WKL}$  has been proved by Gherardi and Marcone [38, Theorem 6.7]. The reduction  $\text{WKL} \leq_{\text{W}} \text{lim}$  follows from the uniform version of the low basis theorem (Theorem 3.4) or can easily be proved directly.

Finally,  $\text{C}_{\#\leq 2}$  is the closed choice problem on Cantor space restricted to sets with one or two elements. This problem was introduced and studied extensively by Le Roux and Pauly [73]. The proof of  $\text{SEP}_1 \equiv_{\text{W}} \text{C}_{\#\leq 2}$  is sketched in Proposition 7.2 in the appendix. The fact  $\text{SEP}_1 \leq_{\text{W}} \text{EC}_1$  is stated by Weihrauch in [109, Section 6]. In the same section we also find the following statement (the first separation as conjecture).

**Proposition 4.5** (Weihrauch [109, Section 6]).  $\text{SEP}_1 <_{\text{W}} \text{EC}_1 <_{\text{W}} \text{EC}$ .

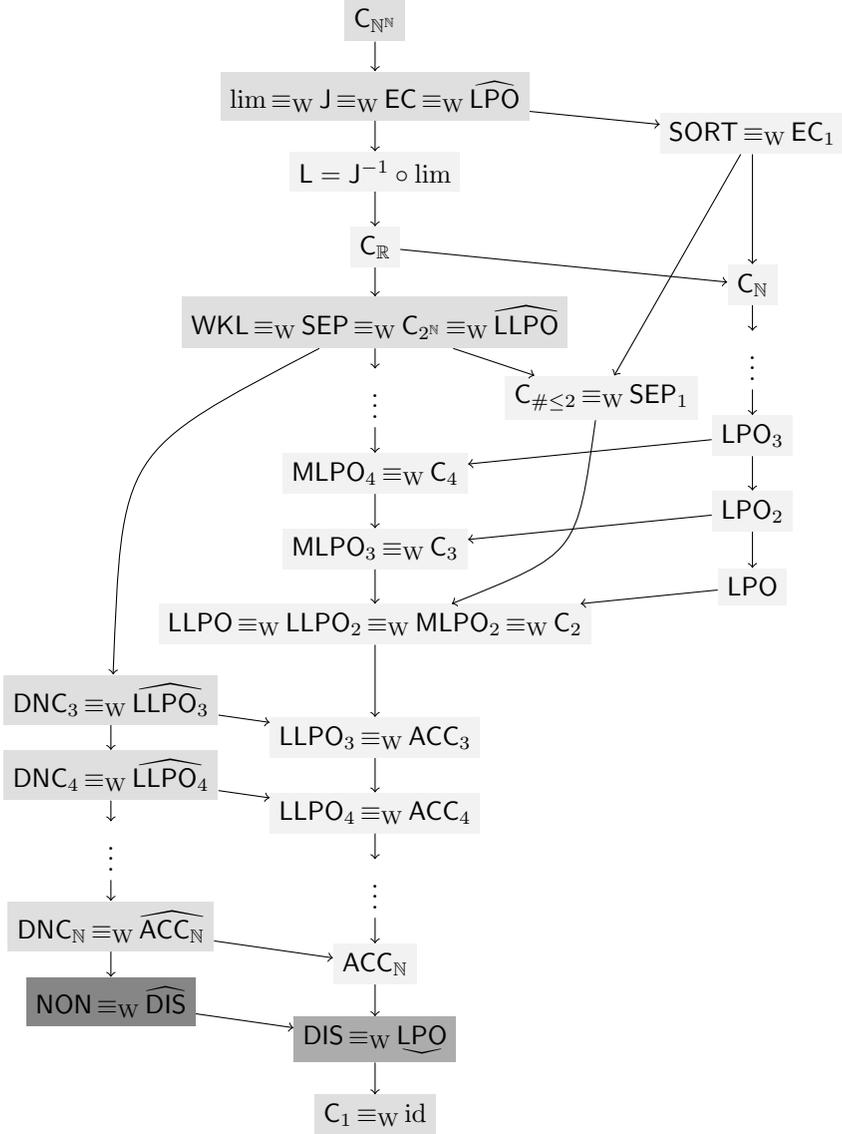


FIGURE 6. Some Weihrauch degrees.

Weihrauch attributes the separation in the second claim to Matthias Schröder. Indeed this follows from the fact that SORT always has a computable output, whereas EC can have non-computable outputs on computable inputs. The other separation  $EC_1 \not\leq_W SEP_1$ , mentioned as conjecture by Weihrauch, can be proved as follows: we have  $LPO \leq_W EC_1$ , but  $LPO \not\leq_W SEP_1$ , since otherwise  $\lim \equiv_W \widehat{LPO} \leq_W WKL$  would follow, which is wrong by the low basis theorem (Theorem 3.4). The details and notations used in this proof will be discussed in the next section.

The diagram in Figure 6 shows a portion of the Weihrauch lattice that includes the problems discussed here together with problems mentioned in the next section. Any arrow indicates a Weihrauch reduction against the direction of the arrow (which is natural, as arrows correspond to logical implications in this way). The diagram should be complete up to transitivity and reflexivity.

## 5 Degrees of omniscience principles

In constructive mathematics principles of omniscience represent certain unsolvable problems of different degrees of complexity. Bishop introduced the *limited principle of omniscience* LPO and the *lesser limited principle of omniscience* LLPO [4]. Richman [94] generalized the latter mentioned principle to principles  $LLPO_n$ . Weihrauch [110] translated these principles into problems in his lattice structure. We formulate them as multi-valued maps for all  $n \geq 1$ , where  $\langle \rangle$  denotes some standard tupling function on Baire space:

$$\begin{aligned} LPO_n &: \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}, \langle p_1, \dots, p_n \rangle \mapsto |\{i \in \mathbb{N} : p_i = \widehat{0}\}|, \\ LLPO_n &: \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}, \langle p_1, \dots, p_n \rangle \mapsto \{i \in \mathbb{N} : p_i = \widehat{0}\} \\ &\quad \text{with } \text{dom}(LLPO_n) := \{\langle p_1, \dots, p_n \rangle : |\{i \in \mathbb{N} : p_i \neq \widehat{0}\}| \leq 1\}, \\ MLPO_n &: \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}, \langle p_1, \dots, p_n \rangle \mapsto \{i \in \mathbb{N} : p_i = \widehat{0}\} \\ &\quad \text{with } \text{dom}(MLPO_n) := \{\langle p_1, \dots, p_n \rangle : |\{i \in \mathbb{N} : p_i = \widehat{0}\}| \geq 1\}. \end{aligned}$$

Using these general problems we define  $LPO := LPO_1$  and  $LLPO := LLPO_2 = MLPO_2$ . Among other results Weihrauch proved the following properties of these problems.

**Theorem 5.1** (Weihrauch [110, Theorems 4.2, 4.3, 5.4]). For all  $n, k \geq 1$  we obtain

1.  $LPO_n <_W LPO_{n+1}$ ,
2.  $LLPO_{n+2} <_W LLPO_{n+1} <_W LPO$ ,
3.  $MLPO_{n+1} <_W MLPO_{n+2}$ ,

4.  $\text{MLPO}_{n+1} <_{\text{W}} \text{LPO}_n$ ,
5.  $\text{LPO}_k \not\leq_{\text{W}} \text{MLPO}_{n+1}$ .

The diagram in Figure 6 illustrates these results. Later on, it was noticed that some of these omniscience problems can also be seen as choice problems. By  $\text{C}_n$  we denote closed choice as defined in the previous section for the space  $n = \{0, 1, \dots, n-1\}$ . By  $\text{ACC}_n$  we denote the *all-or-co-unique choice problem*, which is  $\text{C}_n$  restricted to sets  $A \subseteq n$  with  $|n \setminus A| \leq 1$ . This definition is used analogously for  $\text{C}_{\mathbb{N}}$  and  $\text{ACC}_{\mathbb{N}}$  and the space  $\mathbb{N} = \{0, 1, 2, \dots\}$  of natural numbers. The following was observed in [12, Example 3.2] and [18, Fact 3.3].

**Proposition 5.2** (Choice and omniscience). We obtain  $\text{MLPO}_n \equiv_{\text{W}} \text{C}_n$  and  $\text{LLPO}_n \equiv_{\text{W}} \text{ACC}_n$  for all  $n \geq 2$ .

Some of the reductions and separations shown in Figure 6 have been proved elsewhere. We mention an example on choice and cardinality proved by Le Roux and Pauly [73, Figure 1].

**Proposition 5.3** (Le Roux and Pauly [73]). We obtain

1.  $\text{C}_2 <_{\text{W}} \text{C}_{\#\leq 2} \not\leq_{\text{W}} \text{C}_{\mathbb{N}}$ ,
2.  $\text{C}_3 \not\leq_{\text{W}} \text{C}_{\#\leq 2}$ .

Other interesting results are related to the parallelization of omniscience problems. For any problem  $f : \subseteq X \rightrightarrows Y$

$$\widehat{f} : \subseteq X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}}, (x_n)_n \mapsto \prod_{n \in \mathbb{N}} f(x_n)$$

is called the *parallelization* of  $f$ . This concept was introduced in [14], and it was proved that  $f \mapsto \widehat{f}$  is a closure operator in the Weihrauch lattice. The following result characterizes the parallelizations of LPO and LLPO.

**Proposition 5.4** (B. & Gherardi [14, Theorem 8.2, Lemma 6.3]). We have:  $\widehat{\text{LLPO}} \equiv_{\text{W}} \text{WKL}$  and  $\widehat{\text{LPO}} \equiv_{\text{W}} \text{EC}$ .

It was discovered by Higuchi and Kihara [54] that the parallelization of  $\text{LLPO}_n$  is related to the problems of *diagonally non-computable functions*

$$\text{DNC}_X : \mathbb{N}^{\mathbb{N}} \rightrightarrows X^{\mathbb{N}}, p \mapsto \{f \in X^{\mathbb{N}} : f \text{ is diagonally non-comp. relative to } p\},$$

defined for all  $X \subseteq \mathbb{N}$ . As usual,  $f : \mathbb{N} \rightarrow X$  is called *diagonally non-computable relative to*  $p$ , if  $(\forall i) \varphi_i^p(i) \neq f(i)$ , where  $\varphi_i^p$  is a Gödel numbering of the functions that are computable relative to  $p \in \mathbb{N}^{\mathbb{N}}$ . We now obtain the following parallelizations.

**Proposition 5.5** (Higuchi & Kihara [54, Proposition 81] and B., Hendtlass and Kreuzer [18, Theorem 5.2]).  $\text{DNC}_n \equiv_W \widehat{\text{LLPO}}_n$  for all  $n \geq 2$  and  $\text{DNC}_{\mathbb{N}} \equiv_W \widehat{\text{ACC}}_{\mathbb{N}}$ .

It is interesting to note that Jockusch [59] proved in 1989 a statement that is very similar to Weihrauch's result  $\text{LLPO}_{n+3} <_W \text{LLPO}_{n+2}$  from Theorem 5.1 for the unparallelized case. Jockusch result was expressed in terms of Medvedev reducibility, but the separations imply separations of the corresponding Weihrauch degrees.

**Proposition 5.6** (Jockusch [59, Theorem 6]). We obtain for all  $n \in \mathbb{N}$ :  $\text{DNC}_{\mathbb{N}} <_W \text{DNC}_{n+3} <_W \text{DNC}_{n+2}$ .

In fact, the results of Jockusch and Weihrauch can be jointly generalized to  $\text{ACC}_{n+2} \not\leq_W \text{DNC}_{n+3}$  [18, Proposition 5.7].

For completeness, we have also added to Figure 6 the lowest known natural discontinuous Weihrauch degree, namely that of the *discontinuity problem* DIS [11]. This problem can be defined by *stashing*  $\widehat{\text{LPO}}$  of LPO [10] (stashing is an operation dual to parallelization). We mention that the *non-computability problem* NON, which was introduced and studied in [18], and maps every  $p \in \mathbb{N}^{\mathbb{N}}$  to some  $q \in \mathbb{N}^{\mathbb{N}}$  that is not computable from  $p$ , is the parallelization of DIS [10]. We omit the definitions here and point the interested reader to the references.

The main purpose of the diagram in Figure 6 is to illustrate that some of the problems discussed by Weihrauch in his initial two reports are related to important cornerstones among the Weihrauch degrees, some of which were much later rediscovered and studied under different names. All parallelizable and stashable degrees in Figure 6 are displayed in very dark grey, all only parallelizable degrees are displayed in mid grey, all only stashable degrees in dark grey, and all other degrees in light grey.

## 6 Some work by students in Hagen

The purpose of this section is to briefly discuss some follow-up work on Weihrauch's initial reports [109, 110]. In fact, Weihrauch supervised six M.Sc. and Ph.D. theses on topics related to his reducibility over a period of 18 years. The first of these even appeared before [109, 110] and are listed here in chronological order. Like [109, 110], most of this work remained unpublished for quite some time, and we give some references to later publications that overlap with the content of these theses.

*Diplomarbeit* of Torsten von Stein [106]. In this thesis, the concept of Weihrauch reducibility appears in writing for the first time and the author introduces the so-called C-hierarchy that measures the complexity of a problem  $f$  by the number of applications of the problem EC that are required

to solve  $f$ . He also studies quite a number of specific problems from analysis, most notably the differentiation problem (see Theorem 6.1 below). The results remain unpublished except for some pointers to this work elsewhere.

*Diplomarbeit* of Uwe Mylatz [81]. Uwe Mylatz continued the work of von Stein, and he studied also the second and third level of the C-hierarchy. Among other problems, he looked at higher derivatives, rationality of reals, monotonicity of sequences, convergence, density etc. This work remained unpublished.

*Diplomarbeit* of the present author [5]. In this thesis, it is essentially proved that the C-hierarchy is identical to the effective Borel hierarchy. A much extended version of this thesis was later published in [7], which also includes a proof that for admissible representations of computable Polish spaces Borel measurability on a certain finite level corresponds to the corresponding measurability of the realizers.

Ph.D. thesis of Peter Hertling [48]. The author introduces methods of combinatorial nature that allow to classify the (strong) topological Weihrauch degrees of certain problems with finite or countable discrete image, using forests of trees. These trees essentially capture the nature of the occurring discontinuities. Some of these results were already touched upon in the technical reports [47, 45, 46, 52] and the proceedings article [51]. A significant extension of some of the crucial results in [48] was finally published in [49]. This thesis has also influenced later work on this topic, for instance [97, 96, 50, 65].

Ph.D. thesis of Uwe Mylatz [82]. In his Ph.D. thesis, the author continues the study of the problems  $\text{LPO}_n$ ,  $\text{LLPO}_n$ ,  $\text{MLPO}_n$  and many variants thereof of combinatorial nature. Partially the methods developed in [48] are considered for separation results. The thesis remains unpublished.

M.Sc. thesis of Arno Pauly [88]. In this thesis, the author studies the classification of a number of concrete problems and the structure of the lattice induced by the continuous version of Weihrauch reducibility from a lattice-theoretic perspective. The publication [90] extends many of the structural results.

Presumably, the first time Weihrauch reducibility appeared in a published article was in [6], where the author generalized Pour-El and Richards first main theorem [92]. For instance the fact that the operator  $d$  of differentiation is linear, has a closed graph and satisfies some minimal computability properties implies already the direction  $\text{EC} \leq_W d$  in the following result.

**Theorem 6.1** (von Stein [106, Theorem 12]). Let  $d$  be the operator of differentiation  $d : \subseteq \mathcal{C}[0, 1] \rightarrow \mathcal{C}[0, 1], f \mapsto f'$ , restricted to continuously differentiable functions. Then  $\text{EC} \equiv_W d$ .

## 7 Conclusion

The purpose of these notes is to show how the subject of Weihrauch complexity emerged in work of the Hagen school of computable analysis, in particular, from Weihrauch's original contributions to this topic and work produced by his students. Many more recent developments and prominent degrees were already anticipated in this early work.

Today Weihrauch complexity is a mature research topic that has found interest from researchers in computability theory, computable analysis, reverse mathematics, and proof theory. This is witnessed by two Dagstuhl seminars in 2015 and 2018 that were dedicated to Weihrauch complexity and a number of international Ph.D. and M.Sc. theses that are either mostly or partially dedicated to studying Weihrauch complexity. This includes the theses by Gherardi [37], Pauly [91], Higuchi [53], Carroy [27], Neumann [83, 84], Rakotoniaina [93], Borges [29], Patey [87], Sovine [99], Nobrega [86], Thies [100], Uftring [103], Goh [39], Anglès d'Auriac [1], and Valenti [105] (in chronological order from 2011 onwards, not mentioning the theses discussed in the previous section). A comprehensive up-to-date bibliography on Weihrauch complexity can be found online.

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## Appendix: Some Proofs

In this appendix we add some proofs that justify some claims made earlier. It seems that most of the corresponding equivalences stated here have not been noticed widely and do not appear in writing elsewhere.

We recall that the *sorting problem*  $\text{SORT} : \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}}$  is defined by

$$\text{SORT}(p) := \begin{cases} 0^n \hat{1} & \text{if } p \text{ contains exactly } n \text{ zeros and} \\ \hat{0} & \text{if } p \text{ contains infinitely many zeros.} \end{cases}$$

This problem was introduced by Pauly and Neumann [85]. Here  $\hat{n} = nnn\dots \in \mathbb{N}^{\mathbb{N}}$  denotes the constant sequence with value  $n \in \mathbb{N}$ .

We define the *rationality problem*  $\text{RAT} : \mathbb{R} \rightrightarrows \{0, 1\}^{\mathbb{N}}$  by

$$\text{RAT}(x) := \begin{cases} \{0^n \hat{1} : x = q_n\} & \text{if } x \in \mathbb{Q} \text{ and} \\ \{\hat{0}\} & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

where  $(q_n)_{n \in \mathbb{N}}$  is the standard enumeration of the rational numbers  $\mathbb{Q}$  given by  $q_{\langle n, k, m \rangle} := \frac{n-k}{m+1}$ , where  $\langle \rangle$  denotes a standard Cantor tupling on  $\mathbb{N}$ . That is,  $\text{RAT}$  makes the property of being a rational number c.e. and on top of it, it determines the value of the rational number in the case of a rational input  $x \in \mathbb{R}$ . For irrational inputs  $x$  no information is provided. Since the representations of the reals below and including  $\rho$  in the diagram in Figure 1 are all computably equivalent, if restricted to irrational numbers, it is exactly the problem  $\text{RAT}$  that can help to translate the representations into each other. We prove the following statement about the equivalence class of  $\text{RAT}$ .

**Proposition 7.1.**  $\text{EC}_1 \equiv_{\text{W}} \text{RAT} \equiv_{\text{W}} \text{SORT}$ .

*Proof.* We are going to prove  $\text{RAT} \leq_{\text{sW}} \text{EC}_1 \leq_{\text{sW}} \text{SORT} \leq_{\text{W}} \text{RAT}$ .

We start with proving  $\text{RAT} \leq_{\text{sW}} \text{EC}_1$ . Given an input  $x \in \mathbb{R}$  for  $\text{RAT}$ , we start producing an enumeration  $p \in \mathbb{N}^{\mathbb{N}}$  of  $\mathbb{N} \setminus \{0\} = \text{ran}(p - 1)$  on the output side. Simultaneously, we try falsifying  $x = q_0$ . If it turns out that  $x \neq q_0$ , then we enumerate 0 into our set and we switch to producing an enumeration of  $\mathbb{N} \setminus \{m\}$  with some sufficiently large  $m \in \mathbb{N}$  that has not yet been enumerated and such that  $q_m = q_1$ . Simultaneously, we try falsifying  $x = q_m$ . If we continue inductively in this way with some  $m$  with  $q_m = q_2$  in the next step, then for rational input  $x \in \mathbb{Q}$  we produce some enumeration  $p$  of a set  $A = \mathbb{N} \setminus \{m\} = \text{ran}(p - 1)$  with  $q_m = x$  and for irrational  $x$  we produce some enumeration  $p$  of  $A = \mathbb{N} = \text{ran}(p - 1)$ . Hence, given the characteristic function  $\chi_A = \text{EC}_1(p)$  we can compute a point in  $\text{RAT}(x)$  as follows: if  $\chi_A = \hat{1}$ , then  $\hat{0} \in \text{RAT}(x)$  and if  $\chi_A = 1^m 0 \hat{1}$ , then  $0^m \hat{1} \in \text{RAT}(x)$ .

We now prove  $\text{EC}_1 \leq_{\text{sW}} \text{SORT}$ . Given  $p \in \mathbb{N}^{\mathbb{N}}$  that enumerates a set  $A = \text{ran}(p - 1)$  with  $A = \mathbb{N}$  or  $A = \mathbb{N} \setminus \{m\}$  for some  $m \in \mathbb{N}$ , we compute

a sequence  $q \in \{0, 1\}^{\mathbb{N}}$  as follows. Whenever we have found a consecutive segment  $\{0, 1, \dots, k\} \subseteq A$  of length  $k + 1$  in the enumeration  $p$ , then we ensure that the output  $q$  contains exactly  $k + 1$  zeros. As long as no other information is available on the input side, we append digits 1 to the output  $q$ . This algorithm ensures that  $q$  contains infinitely many zeros (and hence  $\text{SORT}(q) = \widehat{0}$ ) if and only if  $A = \mathbb{N}$  and  $\text{SORT}(q) = 0^m 111\dots$  if and only if  $A = \mathbb{N} \setminus \{m\}$ . Hence given  $\text{SORT}(q)$  we can compute the characteristic function  $\chi_A$  of  $A$  as follows: if  $\text{SORT}(q) = \widehat{0}$  then  $\chi_A = \widehat{1}$  and if  $\text{SORT}(q) = 0^m \widehat{1}$  then  $\chi_A = 1^m \widehat{0}$ .

Now we prove  $\text{SORT} \leq_W \text{RAT}$ . Given an input  $p \in \{0, 1\}^{\mathbb{N}}$  we start producing better and better approximations of  $x_0 = \frac{1}{2^{(0,0)}} \in \mathbb{R}$  on the output side. As soon as we find the first digit 0 in  $p$ , we switch to writing better and better approximations of some number  $x_1 = \frac{2m+1}{2^{(1,k)}} \in \mathbb{R}$  on the output side with suitable  $m, k \in \mathbb{N}$ , such that  $x_1$  is compatible with the previous approximations of  $x_0$  and we start with producing an approximation of  $x_1$  that is good enough such that no rational number with denominator smaller than  $2^{(1,k)}$  can satisfy it (which is possible as  $2m+1$  and  $2^{(1,k)}$  are coprime). In general, if we find the  $(n+1)$ -th digit 0 in  $p$ , then we switch to producing approximations of some number  $x_{n+1} = \frac{2m+1}{2^{(n+1,k)}}$  for appropriate  $m, k \in \mathbb{N}$ , such that  $x_{n+1}$  is compatible with the previous approximation of  $x_n$ . Again the first approximation of  $x_{n+1}$  that we produce is good enough such that no rational number with denominator smaller than  $2^{(n+1,k)}$  can satisfy it. If there are infinitely many zeros in  $p$ , then the output converges to  $x := \lim_{n \rightarrow \infty} x_n$ , which must be irrational, since larger and larger denominators are excluded by the conditions above. If there is only a finite number  $j$  of zeros in  $p$ , then the final output is  $x = x_j$  and from  $0^j \widehat{1} \in \text{RAT}(x)$  we can compute  $m, n, k \in \mathbb{N}$  such that  $x = q_i = \frac{2m+1}{2^{(n,k)}}$ , where the numbers  $m, n, k$  are unique since the numerator and denominator are coprime. Given  $x$  and the original input  $p$ , we can now compute  $\text{SORT}(p)$  as follows: we write as many zeros to the output as we can find in  $p$ . Simultaneously, we try to find some  $i$  with  $0^i \widehat{1} \in \text{RAT}(x)$ . If we find such an  $i$ , then we determine  $n$  as above and we extend the output to  $0^n \widehat{1} = \text{SORT}(p)$ . If we never find such an  $i$ , then  $p$  contains infinitely many zeros and the output is  $\widehat{0} = \text{SORT}(p)$ . Q.E.D.

The equivalence  $\text{RAT} \equiv_W \text{SORT}$  is similar to the statement of [85, Theorem 23].<sup>5</sup>

**Proposition 7.2** (Separation and choice).  $C_{\# \leq 2} \equiv_{sW} \text{SEP}_1$ .

*Proof sketch.* Firstly, we note that

$$\text{dom}(\text{SEP}_1) = \{\langle p, q \rangle \in \mathbb{N}^{\mathbb{N}} : 1 \leq |\text{SEP}\langle p, q \rangle| \leq 2\}$$

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<sup>5</sup>However, [85, Theorem 23] does not seem to hold in full generality as stated, and the proof given here fills a gap and is an alternative proof for the case  $X = \mathbb{R}$ .

and negative information on the closed set  $\text{SEP}_1 \langle p, q \rangle \subseteq 2^{\mathbb{N}}$  can be computed from  $p, q$ . This provides the reduction  $\text{SEP}_1 \leq_{\text{sW}} \mathbf{C}_{\#\leq 2}$ .

For the inverse reduction, we have given by negative information a closed set  $A \subseteq 2^{\mathbb{N}}$  with one or two points. We can compute from the negative information an infinite binary tree  $T$  with  $[T] = A$ . This tree has one or two infinite paths. We now follow the construction in the proof of [38, Theorem 6.7]. There a predicate  $\varphi_T(s, i)$  is used, which indicates that for  $s \in \{0, 1\}^*$  and  $i \in \{0, 1\}$  there is a finite length  $n \in \mathbb{N}$  such that the path  $si$  can be extended in  $T$  up to length  $n$ , but not the path  $s(i-1)$ . Without loss of generality, we assume that words  $s \in \{0, 1\}^*$  are identified with numbers  $s \in \mathbb{N}$  that encode them with respect to some standard encoding. In the proof of [38, Theorem 6.7] it is shown how to compute  $p, q \in \mathbb{N}^{\mathbb{N}}$  such that

$$\begin{aligned} \text{ran}(p-1) &= \{s+2 : \varphi_T(s, 0)\} \cup \{0\}, \\ \text{ran}(q-1) &= \{s+2 : \varphi_T(s, 1)\} \cup \{1\}. \end{aligned}$$

Since our tree  $T$  has exactly one or two infinite paths, it follows that  $\varphi_T(s, 0) \vee \varphi_T(s, 1)$  holds for all  $s$ , except possibly one. That is

$$|\mathbb{N} \setminus (\text{ran}(p-1) \cup \text{ran}(q-1))| \leq 1.$$

Hence,  $\langle p, q \rangle \in \text{dom}(\text{SEP}_1)$ , and the proof of the reduction  $\text{WKL} \leq_{\text{sW}} \text{SEP}$  in [38, Theorem 6.7] extends to a proof of the reduction  $\mathbf{C}_{\#\leq 2} \leq_{\text{sW}} \text{SEP}_1$ .

Q.E.D.