Logics and Rules

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Rosalie Iemhoff

Utrecht University, the Netherlands

In building a foundation for mathematics one develops one big theory, such as set theory, type theory, category theory, ...

In modelling the logical reasoning in a particular setting both expressivity and efficiency may play a role.

Hence the great variety of logics around.

One may wish to establish certain things about these logics: consistency, complexity, conservativity, ...

Good descriptions of a logic can help.

This talk: The possible descriptions of logics.

Areas: mathematics, computer science and philosophy.

Logics (and theories) can be described in various ways: semantically, proof theoretically, ...

Ex. *Classical propositional logic* CPC*consists of all formulas*

- *that evaluate to 1 in all truth tables,*
  (models)
- *that can be derived by Modus Ponens from the axioms ...,*
  (proof systems),
- *that hold in all boolean algebras,*
  (algebras)
- *for which the conjunctive normal form of their negation has a resolution refutation,*
  (proof systems)

$\vdots$

Here we focus on proof–theoretic descriptions.

Inference is the central notion:

For formulas φ *and* ψ*: from* φ *infer* ψ*.*

In most cases one consisders sets of premisses:

For a set of formulas Γ*: from* Γ *infer* φ*.*

Inference is relative to a given logic L*: infer* φ *from* Γ *in* L,

$$\Gamma \vdash_{\mathsf{L}} \varphi.$$

Ex. The $\{\to\}$*-fragment of* CPC *can be described via the Hilbert system* H *consisting of axioms*

$$\varphi \to (\psi \to \varphi) \qquad \big(\varphi \to (\psi \to \chi)\big) \to \big((\varphi \to \psi) \to (\varphi \to \chi)\big)$$

and rule Modus Ponens

$$\frac{\varphi \quad \varphi \to \psi}{\psi}$$

A formula φ *belongs to* CPC *if there are formulas* $\varphi_1, \ldots, \varphi_n = \varphi$ *such that every formula either is an instance of an axiom or follows from earlier formulas by an instance of Modus Ponens.*

Hilbert systems consist of axiom schemes and rule schemes.

Ex. *Natural deduction ND. Consists of axioms and rules such as*

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \qquad \frac{\varphi(y)}{\forall x \varphi(x)} \text{ (y not free in open assumptions)}$$

Ex. *Gentzen calculi GC. The objects are* *sequents*, *expressions* $\Gamma \Rightarrow \Delta$, *where* Γ *and* Δ *are finite* *multisets* *of formulas. Intended interpretation:*

$$I(\Gamma \Rightarrow \Delta) = \bigwedge_{\varphi \in \Gamma} \varphi \to \bigvee_{\psi \in \Delta} \psi.$$

Gentzen calculi consist of rules and axioms such as

$$\Gamma, \varphi \Rightarrow \varphi, \Delta \qquad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}$$

Gentzen calculi are popular proof systems that are a useful tool in the study of logics.

Thm. *ND and GC polynomially simulate each other.*

Dfn. *Given a language and expressions in that language, a rule is an expression of the form* Γ/φ *or* $\frac{\Gamma}{\varphi}$*, where* φ *is an expression and* Γ *a finite set of expressions. It is an axiom if* Γ *is empty.*

Expressions can be formulas, sequents, clauses, equations, . . . From now on, formula stands for all such expressions.

Dfn. *Given a set of rules* L, $\Gamma \vdash_{\mathsf{L}} \varphi$ *iff there are formulas* $\varphi_1, \dots, \varphi_n = \varphi$ *such that every* φ_i *either belongs to* Γ *or there is a rule* Π/ψ *in* L *such that for some substitution* σ*:* $\sigma\psi = \varphi_i$ *and* $\sigma\Pi \subseteq \{\varphi_1, \dots, \varphi_{i-1}\}$.

What a substitution is depends on the context.

Ex. *In propositional logic a substitution is a map from propositional formulas to propositional formulas that commutes with the connectives. If* $\sigma(p) = \neg p$ *and* L *consists of the following rule,*

$$\frac{(\Gamma \Rightarrow p, \Delta) \quad (\Gamma, p \Rightarrow \Delta)}{\Gamma \Rightarrow \Delta}\ \mathit{Cut}$$

then $(\Rightarrow \neg p), (\neg p \Rightarrow) \vdash_{\mathsf{L}} (\Rightarrow)$.

Dfn. *A consequence relation (c.r.) $\vdash$ is a relation between finite sets of formulas and formulas that satisfies*

reflexivity	$\varphi \vdash \varphi$;
monotonicity	$\Gamma \vdash \varphi$ *implies* $\Gamma, \Pi \vdash \varphi$;
transitivity	$\Gamma \vdash \varphi$ *and* $\Pi, \varphi \vdash \psi$ *implies* $\Gamma, \Pi \vdash \psi$;
structurality	$\Gamma \vdash \varphi$ *implies* $\sigma\Gamma \vdash \sigma\varphi$ *for all substitutions* σ.

Thm. *(Łoś & Susko 1957)* *For any set of rules* L, $\vdash_{\mathsf{L}}$ *is a consequence relation, and for every consequence relation* $\vdash$ *there is a set of rules* L *such that* $\Gamma \vdash \varphi$ *if and only if* $\Gamma \vdash_{\mathsf{L}} \varphi$. $\vdash$ *is axiomatized by* L.

Dfn. *Given a set of rules* L, *the set of* *theorems* *of* L *is*

$$Th(\vdash) \equiv \{\varphi \mid \; \emptyset \vdash \varphi \textit{ holds} \}.$$

A conseqence relation $\vdash$ *covers* *a logic if* $Th(\vdash)$ *consists exactly of the formulas that hold in the logic. A set of rules* X *axiomatizes* *a logic if the conseqence relation* $\vdash_{\mathsf{X}}$ *covers it.*

Ex. *The* $\{\to\}$*-fragment of* CPC *is axiomatized by the set of rules* H *consisting of the rule Modus Ponens and the following axioms*

$$\varphi \to (\psi \to \varphi) \qquad (\varphi \to (\psi \to \chi)) \to \big((\varphi \to \psi) \to (\varphi \to \chi)\big).$$

Ex. LK *and* LK $-$ $\{Cut\}$ *both axiomatize the sequent version of* CQC.

Aim *Describe all possible consequence relations that cover a given logic.*

Dfn. Γ/φ *is* *derivable* *iff* $\Gamma \vdash_{\mathsf{L}} \varphi$.

Γ/φ *is* *strongly derivable* *iff* $\vdash_{\mathsf{L}} \bigwedge \Gamma \to \varphi$.

Note $\mathsf{CPC}_{\to}$ *is axiomatized by the set of rules* H. *For all sets* X *of implicational formulas:*

$$\mathsf{H} \cup \mathsf{X} \textit{ covers } \mathsf{CPC}_{\to} \textit{ iff } \mathsf{X} \textit{ consists of tautologies.}$$

Do we have that for all sets X *of implicational rules:*

Question H∪X *covers* $\mathsf{CPC}_{\to}$ *iff* X *consists of rules strongly derivable in* $\mathsf{CPC}_{\to}$*?*

$$Th(\mathsf{H} \cup X) = \mathsf{CPC}_{\to} \quad \textit{iff} \quad \bigwedge \Gamma \to \varphi \textit{ is a tautology for all } \Gamma/\varphi \in \mathsf{X}.$$

Question H∪X *covers* $\mathsf{CPC}_{\to}$ *iff* X *consists of rules derivable in* $\vdash_{\mathsf{H}}$*?*

$$Th(\mathsf{H} \cup X) = \mathsf{CPC}_{\to} \quad \textit{iff} \quad \Gamma \vdash_{\mathsf{H}} \varphi \textit{ for all } \Gamma/\varphi \in \mathsf{X}.$$

Question
What happens with $Th(\vdash)$ is we add rules to the consequence relation?

Dfn. *(Lorenzen '55, Johansson '37)*

$R = \Gamma/\varphi$ *is admissible in* L *iff* $Th(\vdash_{\mathsf{L}}) = Th(\vdash_{\mathsf{L},R})$.

Notation $\Gamma \mathrel{|\!\!\sim}_{\mathsf{L}} \varphi$ *denotes "*Γ/φ *is admissible in* L*".*

Ex. $\varphi(x)/\forall x \varphi(x)$ *admissible in many theories.*

$\bot/\varphi$ *is admissible in any consistent logic, but not always derivable.*

$\varphi/\Box\varphi$ *and* $\Box\varphi/\varphi$ *are admissible in many modal logics.*

Cut is admissible in $\mathsf{LK} - \{Cut\}$ *and shortens proofs superexponentially.*

Note *For all logics* L*:* $\mathrel{|\!\!\sim}_{\mathsf{L}}$ *is a consequence relation and*

$$\Gamma \vdash_{\mathsf{L}} \varphi \;\Rightarrow\; \Gamma \mathrel{|\!\!\sim}_{\mathsf{L}} \varphi \qquad Th(\vdash_{\mathsf{L}}) = Th(\mathrel{|\!\!\sim}_{\mathsf{L}}).$$

Note *The minimal consequence relation* $\vdash$ *for which* $Th(\vdash_{\mathsf{L}}) = Th(\vdash)$ *is*

$$\{\Gamma \vdash \varphi \mid \ \vdash_{\mathsf{L}} \varphi \text{ or } \varphi \in \Gamma\ \}.$$

The maximal consequence relation $\vdash$ *for which* $Th(\vdash_{\mathsf{L}}) = Th(\vdash)$ *is* $\mathrel{|\!\sim}_{\mathsf{L}}$.

Aim *Describe the admissible rules,* $\mathrel{|\!\sim}_{\mathsf{L}}$*, of a given logic* L.

Lemma $\Gamma \mathrel{|\!\sim}_{\mathsf{L}} \varphi$ *iff for all substitutions* σ*:* $\vdash_{\mathsf{L}} \bigwedge \sigma\Gamma$ *implies* $\vdash_{\mathsf{L}} \sigma\varphi$.

Thm. *All admissible rules of* CPC *are strongly derivable.*

Prf. *If* φ/ψ *is admissible, then for all substitutions* σ *to* $\{\top, \bot\}$*: if* $\vdash_{\mathsf{CPC}} \sigma\varphi$*, then* $\vdash_{\mathsf{CPC}} \sigma\psi$*.*

Thus $\varphi \to \psi$ *is true under all valuations. Hence* $\vdash_{\mathsf{CPC}} \varphi \to \psi$*.*

Many many nonclassical logics have nonderivable admissible rules.

Thm. *The Kriesel–Putnam rule*

$$\frac{\neg\varphi \to \psi \vee \chi}{(\neg\varphi \to \psi) \vee (\neg\varphi \to \chi)}\ \mathsf{KP}$$

is admissible but not strongly derivable in intuitionistic logic IQC, *as*

$$(\neg\varphi \to \psi \vee \chi) \to (\neg\varphi \to \psi) \vee (\neg\varphi \to \chi)$$

is not derivable in IQC. *The same holds for Heyting Arithmetic.*

Thm. *(Prucnal '79)*
KP *is admissible in any intermediate logic.*

Thm. *(Buss & Mints & Pudlak '01)*
KP *does not shorten proofs more than polynomially.*

Thm. *The Skolem rule* $\exists x \forall y \varphi(x,y) / \exists x \varphi(x, fx)$ *is admissible but not derivable in (theories in) classical predicate logic* CQC.

Thm. *(Avigad '03)*
If a theory can code finite functions, then the Skolem rule cannot shorten proofs more than polynomially.

Thm. *(Baaz & Hetzl & Weller '12)*
In the setting of sequent calculi and cut-free proofs, the Skolem rule exponentially shortens proofs.

Ex. *If* $\vdash_{\mathsf{KT}} \varphi \to \Box\varphi$ *then* $\vdash_{\mathsf{KT}} \varphi$ *or* $\vdash_{\mathsf{KT}} \neg\varphi$ *(Williamson '92).*

If $\vdash_{\mathsf{IQC}} \varphi \vee \psi$ *then* $\vdash_{\mathsf{IQC}} \varphi$ *or* $\vdash_{\mathsf{IQC}} \psi$.

To express such inferences, extend the notion of consequence to multi-conclusion consequence relations.

Dfn. *A multi-conclusion consequence relation (m.c.r.) $\vdash$ is a relation on finite sets of formulas that satisfies*

- reflexivity $\varphi \vdash \varphi$;
- monotonicity $\Gamma \vdash \Delta$ *implies* $\Gamma, \Pi \vdash \Delta, \Sigma$;
- transitivity $\Gamma \vdash \varphi, \Delta$ *and* $\Pi, \varphi \vdash \Sigma$ *implies* $\Gamma, \Pi \vdash \Delta, \Sigma$;
- structurality $\Gamma \vdash \Delta$ *implies* $\sigma\Gamma \vdash \sigma\Delta$ *for all substitutions* σ.

Dfn. $\Gamma \mathrel{|\!\!\sim}_{\mathsf{L}} \Delta$ *iff for all* σ: $\vdash_{\mathsf{L}} \bigwedge \sigma\Gamma$ *implies* $\vdash_{\mathsf{L}} \sigma\varphi$ *for some* $\varphi \in \Delta$.

Γ/Δ *is derivable if* $\Gamma \vdash_{\mathsf{L}} \varphi$ *for some* $\varphi \in \Delta$.

Note L *has the disjunction property iff* $\varphi \vee \psi \mathrel{|\!\!\sim}_{\mathsf{L}} \{\varphi, \psi\}$.

Aim *Describe both the single-conclusion and multi-conclusion admissible rules of a given logic, via an algorithm or in some other useful way.*

Note *If* $\varphi \mathrel{|\!\sim}_{\mathsf{L}} \psi$, *then* $\varphi \wedge \chi \mathrel{|\!\sim}_{\mathsf{L}} \psi \wedge \chi$.

Dfn. *A set of rules* $\mathcal{R}$ *derives a rule* Γ / Δ *if* $\Gamma \vdash_{\mathsf{L},\mathcal{R}} \Delta$.

$\mathcal{R}$ *is a basis for the admissible rules of* L *iff the rules in* $\mathcal{R}$ *are admissible in* L *and* $\mathcal{R}$ *derives all admissible rules of* L.

Sub aim *Provide a "nice" basis for the single-conclusion and multi-conclusion admissible rules of a given logic.*

Thm. CPC *is structurally complete (all admissible rules are derivable).*

Thm. *(Chagrov '92)*
There are decidable logics in which admissibility is undecidable.

Thm. *(Rybakov '80s) Admissibility is decidable in* IPC *and many modal logics. Admissibility in* IPC *has no finite bases.*

Thm. *(Jeřábek '07)*
In IPC *and many modal logics admissibility is coNEXP-complete.*

Thm. *(Iemhoff '01 & Rozière '95)*
The Visser rules are a basis for the admissible rules of IPC.

Thm. *(Iemhoff '05)*
The Visser rules are a basis for the admissible rules in all intermediate logics in which they are admissible.

Thm. *(Goudsmit&Iemhoff '14)*
The nth Visser rule is a basis for the admissible rules in the $(n+1)$*th Gabbay-De Jongh logic of* $(n+1)$*-branching trees.*

Dfn. *Two sets of rules, in sequent notation:*

$$\frac{\Box\Gamma \Rightarrow \Box\Delta}{\{\boxdot\Gamma \Rightarrow p \mid p \in \Delta\}}\ \mathsf{V}^\bullet \qquad \frac{\Box\Gamma \equiv \Gamma \Rightarrow \Box\Delta}{\{\boxdot\Gamma \Rightarrow p \mid p \in \Delta\}}\ \mathsf{V}^\circ$$

Ex. *For* $\mathsf{L} \in \{\mathsf{K4}, \mathsf{S4}, \mathsf{GL}\}$*:* $\Box q \mathrel{|\!\sim}_{\mathsf{L}} q$ *and* $\Box q \vee \Box r \mathrel{|\!\sim}_{\mathsf{L}} q, r$.

Thm. *(Jeřábek '05)* *The set of rules* $\mathsf{V}^\bullet$ *is a basis for the admissible rules in any* $\mathsf{L} \supseteq \mathsf{GL}$ *in which it is admissible. Similarly for* V° *and* $\mathsf{S4}$*. More results . . .*

Thm. *(Rybakov & Odintsov & Babenyshev '00's)*
Admissibility is decidable in many temporal logics.

Thm. *(Mints '76)*
In IPC*, all nonderivable admissible rules contain* $\vee$ *and* $\rightarrow$*.*

Thm. *(Prucnal '83)*
$\mathrm{IPC}_{\rightarrow}$ *is structurally complete, as is* $\mathrm{IPC}_{\rightarrow,\wedge}$*.*

Thm. *(Cintula & Metcalfe '10)*
The Wroński rules are a basis for the admissible rules of $\mathrm{IPC}_{\rightarrow,\neg}$*.*

For predicate logic there are design choices to be made concerning variables in substitutions.

For some choice, classical predicate logic is structurally complete (all admissible rules are derivable), just like CPC*.*

Thm. (Visser '99) The propositional admissible rules of Heyting Arithmetic and IPC *are equal.*

Thm. (Visser '02)
Admissibility for predicate rules is Π_2*-complete in Heyting Arithmetic.*

The proofs of many of the results above have unification as a key ingredient.

The study of substitutions σ such that $\sigma s =_{\mathsf{E}} \sigma t$.

In a logic L*:* $\varphi =_E \psi$ *is* $\vdash_{\mathsf{L}} \varphi \leftrightarrow \psi$.

Unification theory in logic is the study of substitutions σ *such that* $\vdash_{\mathsf{L}} \sigma\varphi$.

Ex. *In classical propositional logic, if A is satisfiable, it is unifiable (by a substitution that maps every atom to* $\top$ *or* $\bot$*).*

Another unifier of $\varphi = (p \to q)$ *is* $\sigma(r) = \varphi \wedge r$.

Unifier because $\vdash \varphi \wedge p \to \varphi \wedge q$.

Moreover, $\varphi \vdash r \leftrightarrow \sigma(r)$*, and if* $\vdash \tau\varphi$ *then* $\vdash \tau(r) \leftrightarrow \tau\sigma(r)$.

Dfn. σ *is a unifier of* φ *iff* $\vdash \sigma\varphi$.

$\tau \leqslant \sigma$ *iff for some* τ' *for all atoms* p: $\vdash \tau(p) \leftrightarrow \tau'\sigma(p)$.

σ *is a maximal unifier (mu) of* φ *if among the unifiers of* φ *it is maximal.*

A unifier σ *of* φ *is a mgu if* $\tau \leqslant \sigma$ *for all unifiers* τ *of* φ.

A unifier σ *of* φ *is projective if for all atoms* p:

$$\varphi \vdash p \leftrightarrow \sigma(p).$$

A formula φ *is projective if it has a projective unifier (pu).*

Note *Projective unifiers are mgus:* $\vdash \tau\varphi$ *implies* $\vdash \tau(p) \leftrightarrow \tau\sigma(p)$.

Lemma *If φ is projective, then*

$$\varphi \mid\!\sim \psi \;\Leftrightarrow\; \varphi \vdash \psi.$$

Lemma *If for every φ there is a finite set of projective formulas Π_φ and a set of admissible rules $\mathcal{R}$ such that*

$$\bigvee \Pi_\varphi \mid\!\sim \varphi \vdash_{\mathcal{R}} \bigvee \Pi_\varphi,$$

then $\mathcal{R}$ is a basis for the admissible rules of L, *and* L *has finitary unification if it has a disjunction property.*

Prf. *$\varphi \mid\!\sim \psi$ implies $\chi \vdash \psi$ for all $\chi \in \Pi_\varphi$. Hence $\varphi \vdash_{\mathcal{R}} \psi$.*

Dfn. L *has finitary unification if every unifiable formula has finitely many mus, and projective unification if all formulas have projective unifiers.*

Note *If in a logic with projective unification every formula is unifiable, then the logic is structurally complete.*

Ex. $\Box p \vee \Box\neg p$ *has no mgu as* $\tau_0(p) = \top$ *and* $\tau_1(p) = \bot$ *are both unifiers.*

For every unifier σ *of* $\Box p \vee \Box\neg p$, $\sigma \leqslant \tau_0$ *or* $\sigma \leqslant \tau_1$.

Therefore τ_0 *and* τ_1 *are the mus of* $\Box p \vee \Box\neg p$.

A projective approximation of $\Box p \vee \Box\neg p$ *is* $\{\boxdot p, \boxdot\neg p\}$.

Thm. CPC *has projective unification and is structurally complete.*

Thm. (Rybakov '80s) Admissibility is decidable in IPC *and many modal logics. Admissibility in* IPC *has no finite bases.*

Thm. (Ghilardi '99 & Rozière '95)
IPC *has finitary unification.*

Thm. (Iemhoff '05)
The Visser rules are a basis for the admissible rules in all intermediate logics in which they are admissible.

It follows from results by Jeřábek and me that any intermediate logic has a bases for admissibility that consists of rules derivable from the Visser rules.

Thm. (Ghilardi '99, Iemhoff '05) KC ($\neg\varphi \vee \neg\neg\varphi$) *has unitary unification and the Visser rules are a basis for its admissible rules.*

Thm. (Wroński '08) L *has projective unification iff* $\mathsf{L} \supseteq \mathsf{LC}$.

$$\mathsf{LC} \quad (\varphi \to \psi) \vee (\psi \to \varphi)$$

Thm. K4, S4, GL *and many other transitive modal logics have finitary unification (Ghilardi '01).*

Thm. (Jeřábek '05) The set of rules $\mathsf{V}^\bullet$ *is a basis for the admissible rules in any* $\mathsf{L} \supseteq \mathsf{GL}$ *in which it is admissible. Similarly for* V° *and* S4*.*

Thm. (Dzik & Wojtylak '11) $\mathsf{L} \supseteq \mathsf{S4}$ *has projective unification iff* $\mathsf{L} \supseteq \mathsf{S4.3}$*.*

$$\mathsf{S4.3} \quad \Box(\Box\varphi \to \Box\psi) \vee \Box(\Box\psi \to \Box\varphi)$$

Thm. *(Mints '76)*
In IPC*, all nonderivable admissible rules contain* $\vee$ *and* $\rightarrow$*.*

Thm. *(Prucnal '83)*
$\mathsf{IPC}_{\rightarrow}$ *is structurally complete, as is* $\mathsf{IPC}_{\rightarrow,\wedge}$*.*

Thm. *(Cintula & Metcalfe '10)*
$\mathsf{IPC}_{\rightarrow,\neg}$ *has finitary unification and the Wroński rules are a basis for the admissible rules of* $\mathsf{IPC}_{\rightarrow,\neg}$*.*

Thm. (Jeřábek '10) Admissibility in Łukasiewicz logic is in PSPACE. Jeřábek provides explicit basis.

Thm. (Marraa&Spada)
Łukasiewicz logic has nullary unification type.

- *Modal logic* K *and description logics.*
- *Predicate logics.*
- *Substructural logics.*
- *Explanation of admissible rules in terms of what a logic is modelling.*
- *And the list goes on*
- *and on*
- *and on . . .*

Finis