Satisfaction in outer models

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Basic notions:

Let M be a transitive model of ZFC. We say that a transitive model of ZFC, N, is an *outer model* of M if $M \subseteq N$ and $\mathrm{ORD} \cap M = \mathrm{ORD} \cap N$. The outer model theory of M is the collection of all formulas with parameters from M which hold in all outer models of M.

For a set M, define Hyp(M) the least transitive admissible set (a model of KP) containing M as an element (Hyp(M) is of the form $L_\alpha(M)$ for some M).

Recall the following Theorem of Barwise:

Theorem (Barwise)

Let V be the universe of sets. Let $M \in V$ be a transitive model of ZFC, and let φ be an infinitary sentence in $L_{\infty,\omega} \cap M$ in the language of set theory. Then for a certain infinitary sentence φ^* in $L_{\infty,\omega} \cap \mathrm{Hyp(M)}$ in the language of set theory, the following are equivalent:

(i) *ZFC $+ \varphi^*$ is consistent.*

(ii) $\mathrm{Hyp}(M) \models$ *"ZFC $+ \varphi^*$ is consistent".*

(iii) *In any universe W with the same ordinals as V which extends V and in which M is countable, there is an outer model N of M, $N \in W$, where φ holds.*

In particular, the set of formulas with parameters in M satisfied in an outer model M in an extension where M is countable is definable in $\mathrm{Hyp(M)}$.

It is instructive to see what φ^* looks like:

$$\varphi^* = ZFC \,\&\, \bigwedge_{x \in M} (\forall y \in \bar{x})(\bigvee_{a \in x} y = \bar{a}) \,\&$$

$$\&\, [(\forall x)(x \text{ is an ordinal} \rightarrow \bigvee_{\beta \in M \cap \mathrm{ORD}} x = \bar{\beta})] \,\&\, \mathrm{AtDiag}(M) \,\&\, \varphi,$$

where AtDiag(M), the atomic diagram of M, is the conjunction of all atomic sentences and their negations which hold in M (when the constants are interpreted by the intended elements of M).

Question: Is it consistent that for some M, the satisfaction in outer models is lightface definable in M?(We call such an M, if it exists, *omniscient*.)

Note that if M is definable in all its generic extensions (such as L, or K for small cardinals), then M cannot be omniscient by undefinability of truth (Tarski).

Seeing that L cannot be omniscient, can M be a model of $V = HOD$ and be omniscient?

With many large cardinals, every M is omniscient:

Theorem (M. Stanley)

Suppose that M is a transitive set model of ZFC. Suppose that in M there is a proper class of measurable cardinals, and indeed this class is Hyp(M)*-stationary, i.e.* $\mathrm{Ord}(M)$ *is regular with respect to* Hyp(M)*-definable functions and this class intersects every club in* $\mathrm{Ord}(M)$ *which is* Hyp(M)*-definable. Then M is omniscient.*

Hint: Consider φ^* and φ^*_κ which are the infinitary sentences which say in Hyp of the relevant structure that there is an outer model of M, or $(V_\kappa)^M$ respectively, κ measurable in M. Then:

(*) φ^* is consistent iff φ holds in an outer model of M iff φ^*_κ are consistent for all κ iff for all κ, φ holds in an outer model of $(V_\kappa)^M$.

Question: Are large cardinals necessary for omniscience?

We show that that no: indeed, one inaccessible is enough to get an omniscient model which moreover satisfies $V = HOD$.

Theorem (Friedman, H.)

Assume $V = L$. Let κ be the least inaccessible, and let $M = L_\kappa$. There is a good iteration $(\mathbb{P}, h)$ in V such that if G is $\mathbb{P}$-generic over V, then for some set $\tilde{G}$, which is defined from G, $M[\tilde{G}]$ is an omniscient model of ZFC. Moreover, $M[\tilde{G}]$ is a model of $V = HOD$.

What is a good iteration?

Assume $V = L$. Let κ be the least inaccessible cardinal and let X be the set of all singular cardinals below κ. Fix a partition $\langle X_i \mid i < \kappa \rangle$ of X into κ pieces, each of size κ, such that $X_i \cap i = \emptyset$ for every $i < \kappa$.

Definition

Let μ be an ordinal less than κ^+. We say that (P, f) is a *good iteration of length* μ if it is an iteration $P_\mu = \langle (P_i, \dot{Q}_i) \mid i < \mu \rangle$ with $< \kappa$ support of length μ, $f : \mu \to X$ is an injective function in L and the following hold:

(i) $\mathrm{rng}(f) \cap X_i$ is bounded in κ for every $i < \kappa$,

(ii) For every $i < \mu$, P_i forces that $\dot{Q}_i$ is either $\mathrm{Add}(f(i)^{++}, f(i)^{+4})$ or $\mathrm{Add}(f(i)^{+++}, f(i)^{+5})$.

Note that $(\mathbb{P}, h)$ from the theorem is an iteration of length κ, composed of good iterations (and hence is equivalent to a good iteration of some length $< \kappa^+$).

The main idea of the proof of the Theorem is as follows:

- We want to decide the membership or non-membership of κ-many formulas with parameters in the outer model theory of the final model. We are going to define an iteration of length κ, dealing with the i-th formula at stage $\mathbb{P}_i$.
- Suppose at stage i, it is possible to kill φ_i by a good iteration $\dot{W}_i$, i.e. ensure that in $V^{\mathbb{P}_i * \dot{W}_i}$ there is no outer model of φ_i. If such $\dot{W}_i$ exists, set $\mathbb{P}_{i+1} = \mathbb{P}_i * \dot{W}_i * \dot{C}_i$, where $\dot{C}_i$ codes this fact by means of a good iteration.

- In the final model $M[\tilde{G}]$, we can decide the membership of φ_i in the outer model theory by asking whether at stage i we have coded the existence a witness $\dot{W}_i$ which kills φ_i.

Hints:

- If there is no outer model of $M[\tilde{G}]$ where φ_i holds, then indeed we have coded this fact at stage i by using some $\dot{W}_i$ (because the tail of $\mathbb{P}$ – itself a good iteration – from stage i did kill φ_i so some such $\dot{W}_i$ must have existed).
- Conversely, if there is an outer model of $M[\tilde{G}]$ where φ_i holds, then we could not have found a witness $\dot{W}_i$ because if we did, then its inclusion in $\mathbb{P}$ would ensure that φ_i is killed.

Note that there is no bound on the length of $\dot{W}_i$, except that it must be less than κ^+ (by the injectivity of the function f which makes $(\dot{W}_i, f)$ a good iteration).

Open questions.

Q1. Suppose M is an omniscient model. Is a set-generic extension of M still omniscient? Or an extension by a Cohen real?

Q2. What is the consistency strength of having an omniscient M? By Theorem, the upper bound is ZFC plus "there is an inaccessible cardinal." Can this be improved to ZFC + "there is a standard model of ZFC"?